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E_{10} orbifolds

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Abstract: We study $\mathbb{Z}_2$ orbifolds of M-theory in terms of E_{10}. We find a simple relation between the $\mathbb{Z}_2$ action on E_{10} and the imaginary root that corresponds [hep-th/0401053] to the "twisted sector" branes. We discuss the connection between the Kac-Moody algebra DE_{10} and the "untwisted" sector, and we demonstrate how DE_{18} can describe both the untwisted and twisted sectors simultaneously.

Keywords: p-branes, String Duality, M-Theory.

Contents

1. Introduction **2**

2. Preliminaries **4**
- 2.1 E_{10} 5
- 2.2 M-theory on T^{10} 6
- 2.3 Roots and fluxes 7

3. The orbifolds **8**
- 3.1 $T^5/\mathbb{Z}_2$ 9
- 3.2 $T^8/\mathbb{Z}_2$ 9
- 3.3 $S^1/\mathbb{Z}_2$ 10
- 3.4 Type IA 10
- 3.5 $T^9/\mathbb{Z}_2$ 11
- 3.6 $T^4/\mathbb{Z}_2$ 12
- 3.7 Summary 12

4. The untwisted sector and DE_{10} **12**
- 4.1 Definition of $\mathfrak{g}^{(\mathrm{inv})}$ 12
- 4.2 The Kac-Moody algebra DE_{10} 13
- 4.3 The subgroup $DE_{10} \subset \mathfrak{g}^{(\mathrm{inv})}$ 14
- 4.4 Comparing DE_{10} to $\mathfrak{g}^{(\mathrm{inv})}$ 15

5. The Kac-Moody algebra $DE_{18(10)}$ **20**
- 5.1 Basic Properties of DE_{18} 21
- 5.2 The Cartan matrix 21
- 5.3 The real form $DE_{18(10)}$ 22
- 5.4 The Subalgebras DE_{10} and $\mathfrak{so}(16)$ 24
- 5.5 The maximal compact subalgebra $\mathfrak{k}(DE_{18(10)})$ 27

6. The twisted sector **27**
- 6.1 $\mathfrak{so}(16)$-packs 27
- 6.2 Examples of $\mathfrak{so}(16)$-packs 29

7. Physical interpretation **30**
- 7.1 Fluxes of the twisted sector 30
- 7.2 $\mathfrak{so}(16)$-packs for Fluxes 31
- 7.3 Imaginary roots 32

8. Type IA **35**

9. The Orbifold $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$ **37**

10. Conclusions and discussion **38**

A. Proof that $DE_{10} \subset \mathfrak{g}^{(\mathrm{inv})}$ **40**

B. Proof that $DE_{10} \subset \mathfrak{g}^{(\mathrm{com})}$ **40**

C. Proof of proposition 6.1 **41**
C.1 Equivalence of root lattices 42

D. Denominator formula for $\mathfrak{g}^{(\mathrm{inv})}$ **43**

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1. Introduction

In this paper we extend a previous work [1] on the connection between M-theory and E_{10}. There is a very intriguing connection between the infinite dimensional Lie group E_{10} and M-theory on T^{10} [2, 3]. Generally speaking, fluxes of M-theory on T^{10} correspond to certain generators, or more precisely *roots*, of E_{10}. Yet, not all generators of E_{10} are accounted for by fluxes. It was proposed in [1] that Kaluza-Klein particles and all the objects that can be obtained from them by U-duality (M2-branes, M5-branes, etc.) can be associated with other generators of E_{10}, which are not associated with fluxes. The interpretation of all generators is not yet complete, and there are an infinite number of E_{10} generators that are associated with neither fluxes nor particles. Nevertheless, the physical interpretation of the E_{10} generators that we have so far is sufficient to start analyzing more complicated backgrounds of M-theory.

In general, starting with an M-theory background X and a discrete symmetry group Γ, we can often construct a new consistent background X/Γ — the Γ-orbifold of X. In this procedure, we keep the Γ-invariant "modes" and discard the "modes" that are not invariant. In addition, consistency requirements such as anomaly cancellation and charge conservation often necessitate the insertion of extra degrees of freedom – the "twisted" sector. In perturbative string theory, there is a well defined procedure to find the twisted sectors [4]. Recent developments in this field appear in [5] (which are the latest in a series of papers starting with [6]). Unfortunately, at the moment there is no general principle to determine the contents of the twisted sector in M-theory; a case-by-case analysis, which sometimes relies on a clever guess, is required.

In this paper we will take X to be T^{10}, and Γ will be some $\mathbb{Z}_2$ action, which needs to be specified. One of the most famous examples of such a $T^{10}/\mathbb{Z}_2$ orbifold is $T^9 \times (S^1/\mathbb{Z}_2)$, for which the twisted sector at low-energy was found by Hořava and Witten [7] to be an $E_8 \times E_8$ 9+1D Super-Yang-Mills theory.

Often, the twisted-sector contains an extra finite number of branes. For example the Hořava -Witten orbifold can be viewed as type-IIA string theory compactified on $S^1/\mathbb{Z}_2$ ($\times T^8$, in our case) with 16 extra (T^8-filling) D8-branes [8]. This setting is known as type-IA. The compactification of M-theory on $T^5/\mathbb{Z}_2$, as another example, was studied in [9][10] and argued in [10] to contain 16 M5-branes. Similarly, M-theory on $T^8/\mathbb{Z}_2$ contains 16 M2-branes [11]–[14], and M-theory on $T^9/\mathbb{Z}_2$ contains 16 units of Kaluza-Klein momentum [9, 12]–[15]. All of these examples are, in fact, dual to each other [9][12].

In this paper, we will re-examine the orbifolds $T^{10}/\mathbb{Z}_2$ and redefine them in terms of E_{10}. We will extend the $\mathbb{Z}_2$ action to an automorphism of the root-lattice of E_{10} and then to an automorphism of the whole E_{10} Lie algebra. We will show that some of the features of the twisted-sector have a simple E_{10}-description. In particular, there is an algebraic connection between the $\mathbb{Z}_2$ action on the Cartan subalgebra of E_{10} and the E_{10}-root that, according to [1], is associated with the branes of the twisted sector. Thus, the details of the $\mathbb{Z}_2$ action algebraically determine the types of branes in the twisted sector.

Naturally, we will refer to the subalgebra of E_{10} that is fixed by (invariant under) the $\mathbb{Z}_2$ automorphism as the *untwisted sector*. We will also refer to the $\mathbb{Z}_2$-invariant sublattice of the E_{10} root lattice as the *untwisted sector of the root lattice*. In appendix C.1 we will demonstrate that for each of the $\mathbb{Z}_2$ orbifolds that we study, the untwisted sector of the root lattice is isomorphic to the root lattice of the infinite dimensional Kac-Moody algebra DE_{10}. We will also show that the untwisted sector of E_{10} contains a DE_{10} subalgebra, but is actually larger than DE_{10}.

The orbifolds that we will study are also dual to heterotic string theory compactified on T^9. This compactification has been studied in [16]. There is reason to suspect that the Kac-Moody algebra DE_{18} appears in this setting. Its exact role is yet unknown, but it is probably similar to the role of E_{10} in toroidal M-theory compactification. DE_{18} is a natural extension of the affine Lie algebra $\widehat{\mathfrak{so}}(24,8)$ that appears when heterotic string theory is compactified on T^8 (see [17]–[19] and references therein).

DE_{10} is obviously a subalgebra of DE_{10+k} for any $k \geq 0$. The Lie algebra DE_{10+k} actually has a $DE_{10} \oplus D_k$ subalgebra, where D_k is isomorphic to the classical Lie algebra $\mathfrak{so}(2k)$. The case $2k = 16$ is physically special because the number 16 that appears here is, as we shall show, directly related to the appearance of 16 branes in the twisted sector of the orbifolds above. (These could, for example, be the 16 M5-branes of the twisted sector of $T^5/\mathbb{Z}_2$.) This extra $\mathfrak{so}(16)$ subalgebra will allow us to group the generators of DE_{18} in $\mathfrak{so}(16)$-representations. We will propose that if a generator belongs to a nontrivial

$\mathfrak{so}(16)$-representation it is related to the twisted sector, and the details of the representation determine the relationship between the generator and the 16 branes of the twisted sector. For example, a DE_{18} generator belonging to the fundamental representation is associated with a single twisted-sector brane. We will present a few examples to support this proposal.

We have to mention that there is yet another infinite dimensional Kac-Moody Lie algebra that is believed to be associated with heterotic string theory compactified on T^9. This is the non-simply-laced BE_{10}, and it was shown in [20] that it is associated to the cosmological evolution of the toroidally compactified heterotic universe. We will not discuss BE_{10} in this paper, since it does not seem to contain the information needed to distinguish between the 16 twisted sectors.

As this work was close to completion, another paper that has some overlap with our work appeared [21]. In this paper the orbifold invariant subalgebra of E_{10} is also identified and it is also noted that it contains a proper DE_{10} subalgebra. This is demonstrated by decomposing under the finite D_9 subalgebra. We had arrived at the same conclusion, however, by decomposing under the affine $\widehat{D}_8$ subalgebra. $\mathbb{Z}_2$-orbifolds of E_{10} and other Kac-Moody algebras have been also recently discussed in [22], with similar conclusions. Other recent ideas related to orbifolds of infinite dimensional "hidden symmetries" of M-theory appear in [23].

The paper is organized as follows. In section 2 we briefly review E_{10} and its relation to M-theory on T^{10}. In section 3 we discuss the $\mathbb{Z}_2$ orbifolds and demonstrate the algebraic connection between the associated $\mathbb{Z}_2$ action on the E_{10} root lattice and the root that is associated with the twisted-sector branes. In section 4 we demonstrate that the $\mathbb{Z}_2$-invariant part of the E_{10} root lattice is isomorphic to the DE_{10} root lattice, and we show that DE_{10} is a proper subalgebra of the $\mathbb{Z}_2$ invariant part of E_{10}. In section 5 we review DE_{18}, and in section 6 we study the algebraic properties of the twisted sector – the generators of DE_{18} that fall into nontrivial $\mathfrak{so}(16)$ representations. In section 7 we connect the physical twisted sector of the orbifold to its algebraic description in terms of DE_{18}. Most of our discussion is presented for the $T^5/\mathbb{Z}_2$ orbifold, but in section 8 we write down, for completeness, the relevant formulas for the dual type-IA string theory. In section 9 we present a preliminary discussion on a more complicated orbifold, $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$, which is dual to M-theory on certain Calabi-Yau 3-folds [24]. We conclude with a discussion in section 10. In appendices A–C we fill-in some technical details about the definitions of the various subalgebras. In appendix C.1 we explicitly show that the $\mathbb{Z}_2$ invariant part of the root lattice of E_{10} is equivalent to the root lattice of DE_{10} and also to the sublattice of DE_{18} that is orthogonal to the roots of $\mathfrak{so}(16) \subset DE_{18}$. In appendix D we present a generating function for the multiplicities of the $\mathbb{Z}_2$ invariant roots of E_{10} that can be compared with a variant of the "denominator formula" for the multiplicities of roots of DE_{10}.

2. Preliminaries

We assume basic familiarity with Kac-Moody algebras. For a thorough treatment see [25].

Given a Kac-Moody Lie algebra X, we use the following notation:

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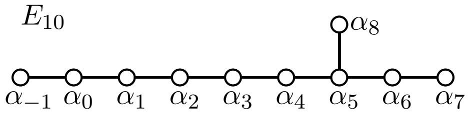

Figure 1: The Dynkin diagram of E_{10}.

$\mathfrak{h}(X) =$	Cartan subalgebra (CSA) of X
$\Delta(X) =$	roots of X
$Q(X) =$	root lattice of X
$P(X) =$	weight lattice of X
$W(X) =$	Weyl group of X
$\mathfrak{g}(X)_\alpha =$	root space (for any root $\alpha \in \Delta(X)$)
$(\alpha\|\beta) =$	The Killing form [for any two roots $\alpha, \beta \in \mathfrak{h}(X)^*$]
$\langle \alpha, h\rangle =$	The inner product [for any root $\alpha \in \mathfrak{h}(X)^*$ and CSA element $h \in \mathfrak{h}(X)$]
$L(\Lambda) =$	Irreducible representation of X with a dominant highest-weight $\Lambda \in P(X)$.

We have the root space decomposition

$$X = \bigoplus_{\alpha\in\Delta(X)} \mathfrak{g}(X)_\alpha, \qquad \mathfrak{g}(X)_\alpha := \{x \in X\colon \quad [h, x] = \langle \alpha, h\rangle x \quad \forall h \in \mathfrak{h}(X)\}\,. \tag{2.1}$$

2.1 E_{10}

The Dynkin diagram of E_{10} is given in figure 1.

It is possible to find a basis of $\mathfrak{h}(X)^*$ in which the root-lattice $Q(E_{10})$ is given by

$$\left\{(n_1, \ldots , n_{10})\colon \quad \sum_{i=1}^{10} n_i \in 3\mathbb{Z}\,, \qquad n_i \in \mathbb{Z} \qquad (i = 1, \ldots , 10)\right\}. \tag{2.2}$$

In this basis a root α is given by a series of 10 integers,

$$\alpha = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}), \tag{2.3}$$

and the square of the root is

$$\alpha^2 = \sum_1^{10} n_i^2 - \frac{1}{9}\Big(\sum_1^{10} n_i\Big)^2\,.$$

The simple roots can be written in this basis as

$$\begin{aligned}
\alpha_{-1} &:= (1, -1, 0, 0, 0, 0, 0, 0, 0, 0)\,,\\
\alpha_0 &:= (0, 1, -1, 0, 0, 0, 0, 0, 0, 0)\,,\\
\alpha_1 &:= (0, 0, 1, -1, 0, 0, 0, 0, 0, 0)\,,\\
\alpha_2 &:= (0, 0, 0, 1, -1, 0, 0, 0, 0, 0)\,,
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 &:= (0,0,0,0,1,-1,0,0,0,0)\,,\\
\alpha_4 &:= (0,0,0,0,0,1,-1,0,0,0)\,,\\
\alpha_5 &:= (0,0,0,0,0,0,1,-1,0,0)\,,\\
\alpha_6 &:= (0,0,0,0,0,0,0,1,-1,0)\,,\\
\alpha_7 &:= (0,0,0,0,0,0,0,0,1,-1)\,,\\
\alpha_8 &:= (0,0,0,0,0,0,0,1,1,1)\,.
\end{aligned}
\tag{2.4}
$$

For the rest of the paper, whenever we express an E_{10} root α as a series of 10 integers, it will be understood to be in the basis (2.4).

We define the *fundamental weights* $\Lambda_i \in \mathfrak{h}(E_{10})^*$ ($i = -1, \ldots, 8$) as the solutions to the linear equations

$$(\Lambda_i|\alpha_j) = \delta_{ij}\,, \qquad i, j = -1, \ldots, 8\,.$$

For future reference, we need the sum of the fundamental weights of E_{10}. This is the weight that enters into character formulas and is given by

$$\rho = -30\alpha_{-1} - 61\alpha_0 - 93\alpha_1 - 126\alpha_2 - 160\alpha_3 - 195\alpha_4 - 231\alpha_5 - 153\alpha_6 - 76\alpha_7 - 115\alpha_8\,.$$

We can also write it as

$$\rho = -(30, 31, 32, 33, 34, 35, 36, 37, 38, 39)\,. \tag{2.5}$$

2.2 M-theory on T^{10}

In the basis dual to (2.4), an element of $\mathfrak{h}(E_{10})$ can be written as

$$\vec{h} = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10})\,,$$

so that

$$\langle \vec{h}, \alpha \rangle = \sum_{i=1}^{10} n_i h_i\,.$$

For M-theory applications, we will set $h_i = \log(M_p R_i)$ ($i = 1, \ldots, 10$), where R_i are called the *compactification radii*, and M_p is the eleven-dimensional Planck mass. $2\pi R_i$ are the sizes of the directions of the T^{10} on which M-theory is compactified. Thus, the connection with M-theory is given by the following identification,

$$\vec{h} = (\log[M_p R_1], \log[M_p R_2], \ldots, \log[M_p R_{10}])\,. \tag{2.6}$$

A classical approximation is valid when $M_p R_i \gg 1$ for all $i = 1, \ldots, 10$. In a dynamical setting, R_i and therefore $\vec{h}$ vary as a function of time, but we will not need to discuss time dependence in the scope of this work.

2.3 Roots and fluxes

A few roots of E_{10} can be associated with zero modes of 10+1D supergravity fields. The bosonic fields of 10+1D supergravity are the metric and 3-form. We denote their (spatial) components by g_{IJ} ($1 \leq I \leq J \leq 10$) and C_{IJK} ($1 \leq I < J < K \leq 10$). We denote the supergravity variable associated with the root α, if it is established, by $\mathcal{C}_\alpha$. For the simple root α_8, for example, we have $\mathcal{C}_{\alpha_8} = C_{89\,10}$, according to equation (2.4). The other simple roots α_i ($i = -1, \ldots, 7$), listed in equation (2.4), are associated with ratios $g_{(i+2),(i+3)}/g_{(i+3),(i+3)}$ of metric components.

Similarly, the field C_{IJK} corresponds to the positive root (2.3) with $n_I = n_J = n_K = 1$, and $n_i = 0$ for $i \neq I, J, K$. The ratio of metric components g_{IJ}/g_{JJ} corresponds to (2.3) with $n_I = 1$, $n_J = -1$ and $n_i = 0$ for $i \neq I, J$.

There is a simple way to see which variable is associated to a given root α. Using the identification (2.6), the expression $2\pi \exp\langle \vec{h}, \alpha\rangle$ can be written as a monomial in the radii $R_1, \ldots, R_{10}$. This monomial can be interpreted as the real part of an action of an instanton. The imaginary part of the action is the variable $\mathcal{C}_\alpha$. For example, $2\pi \exp\langle \vec{h}, \alpha_{-1}\rangle = 2\pi R_1/R_2$ is the real part of the action of a Kaluza-Klein instanton whose imaginary part is $2\pi i g_{12}/g_{22}$. Similarly, $2\pi \exp\langle \vec{h}, \alpha_8\rangle = 2\pi M_p^3 R_8 R_9 R_{10}$ is the real part of the action of an M2-brane instanton. Its imaginary part is $(2\pi)^3 i C_{89\,10}$. (See [26] for more details.)

Any *real root*, i.e. a root α that satisfies $(\alpha|\alpha) \equiv \alpha^2 = 2$, is Weyl-dual to any one of the simple roots. In other words, for any $i = -1, \ldots, 8$, there exists an element w in the Weyl-group of E_{10}, $w \in W(E_{10})$, such that $w(\alpha) = \alpha_i$ [25]. In M-theory, $W(E_{10})$ is a subgroup of the U-duality group [27][28], which is $E_{10}(\mathbb{Z})$ [29]. Thus, any real root α can be associated with an instanton formally [26][30]. If we assume

$$R_1 \gg R_2 \gg \cdots \gg R_{10} \gg \frac{1}{M_p^3 R_8 R_9}\,, \tag{2.7}$$

then whenever α is a positive root $2\pi \exp\langle \vec{h}, \alpha\rangle$ is large, and thus can be interpreted as an instanton action of some object that is formally U-dual to a Kaluza-Klein instanton (or M2-brane). The requirement (2.7) is always satisfied in the classical limit of eleven-dimensional supergravity,

$$R_1 \gg R_2 \gg \cdots \gg R_{10} \gg M_p^{-1}\,. \tag{2.8}$$

Imaginary roots, which satisfy $\alpha^2 \leq 0$, are not Weyl-equivalent (U-dual) to real roots and therefore cannot be interpreted as instantons. It was proposed in [1] that at least a subset of them can be associated with Minkowski brane charges. This subset is the set of positive prime isotropic roots — positive roots that satisfy $\alpha^2 = 0$ and cannot be written as a nontrivial integer multiple of another root.

To find the brane charge that corresponds to a given positive prime isotropic root α, we need to examine the monomial

$$m(\alpha) = \frac{e^{\langle \vec{h}, \alpha\rangle}}{M_p^9 R_1 \cdots R_{10}} \tag{2.9}$$

and identify it with the energy of a Minkowski brane. The denominator of (2.9) comes naturally if we realize that the Weyl-invariant numerator $\exp\langle\vec{h},\alpha\rangle$ measures the energy in units dual to *conformal time* [1], which is usually defined by the differential equation

$$d\tilde{\tau} = \frac{dt}{2\pi M_p^9 R_1(t)\cdots R_{10}(t)}\,, \tag{2.10}$$

where the explicit dependence of radii on time t was restored.

We will now present two examples of positive prime isotropic roots and their associated brane charges. For the first example take

$$\alpha = (0,1,1,1,1,1,1,1,1,1)\,.$$

We get $m(\alpha) = 1/R_1$ which identifies it with a Kaluza-Klein particle. For the second example take

$$\alpha = (2,2,1,1,1,1,1,1,1,1)\,.$$

We get $m(\alpha) = M_p^3 R_1 R_2$, which identifies it with an M2-brane. All positive prime isotropic roots of E_{10} are Weyl-equivalent, and therefore any such root can be identified with a formal U-dual of a Minkowski brane. More details and more arguments in favor of the correspondence between brane charges and imaginary roots can be found in [1].

In [31] a different program for identifying imaginary roots of E_{10} with supergravity fields was suggested. There, imaginary roots were associated with multiple derivatives of supergravity fields. In this paper we will only discuss imaginary roots in the context of their corresponding brane charges, but it would be interesting to extend our discussion of E_{10} orbifolds also to the multiple-derivative fields of [31].

Also, we have to mention a different scheme [32]–[34] in which brane charges are related to real-roots of E_{11}. In this paper we will restrict ourselves to E_{10}. We will also discuss only bosonic excitations. A more complete description probably requires an extension of E_{10} to a superalgebra. A supersymmetric version of E_{11} was recently described in [35].

Finally, we mention a recent interesting idea about yet another connection between E_9 and M-theory [36]–[38]. In this context E_9 is conjectured to be a symmetry of a certain "topological" sector of M-theory. It appears that this E_9 has a different real structure than the one in our context.

3. The orbifolds

We will now begin to study the $\mathbb{Z}_2$-orbifolds. The following definition will be useful.

Definition 3.1. We say that two roots $\alpha, \beta \in \Delta$ are *equivalent mod* 2 if

$$(\alpha|\eta) \equiv (\beta|\eta) \pmod 2$$

for every $\eta \in Q$ (where Q is the root lattice).

For each orbifold we will first identify the $\mathbb{Z}_2$-charge of each supergravity field C_{IJK} and g_{IJ}. This will allow us to identify the $\mathbb{Z}_2$-charge of an arbitrary variable $\mathcal{C}_\alpha$ associated with the positive root α. Next, we will find a relation between the exceptional branes and the $\mathbb{Z}_2$ action. According to [1], the exceptional branes, like any other Minkowski brane, are associated with an imaginary root τ. We will show that the $\mathbb{Z}_2$-charge of any $\mathcal{C}_\alpha$ is determined by τ; the variable $\mathcal{C}_\alpha$ is $\mathbb{Z}_2$-even ($\mathbb{Z}_2$-odd) when $(\alpha|\tau)$ is an even (odd) integer. We will now demonstrate these ideas for various $\mathbb{Z}_2$-orbifolds.

3.1 $T^5/\mathbb{Z}_2$

We start with the orbifold $T^5/\mathbb{Z}_2$. The charge assignments are [9][10]

$$
\begin{aligned}
g_{ij} &\to g_{ij}\,, && 1 \le i,j \le 5\,,\\
g_{iI} &\to -g_{iI}\,, && 1 \le i \le 5\,, \quad 6 \le I \le 10\\
g_{IJ} &\to g_{IJ}\,, && 6 \le I,J \le 10\\
C_{ijk} &\to -C_{ijk}\,, && 1 \le i < j < k \le 5\,,\\
C_{ijK} &\to C_{ijK}\,, && 1 \le i < j \le 5\,, \quad 6 \le K \le 10\\
C_{iJK} &\to -C_{iJK}\,, && 1 \le i \le 5\,, \quad 6 \le J < K \le 10\\
C_{IJK} &\to C_{IJK}\,, && 6 \le I < J < K \le 10\,.
\end{aligned} \tag{3.1}
$$

Every C_{IJK} ($1 \le I < J < K \le 10$) and g_{IJ} ($1 \le I \le J \le 10$) is associated with a positive real root. Using (2.4), we can associate a $\mathbb{Z}_2$-charge with every simple root as follows,

$$
\mathcal{C}_{\alpha_i} \to \mathcal{C}_{\alpha_i} \quad (i \neq 3)\,, \qquad \mathcal{C}_{\alpha_3} \to -\mathcal{C}_{\alpha_3}\,.
$$

We say that the $\mathbb{Z}_2$-charge of $\alpha_{-1}, \ldots, \alpha_2, \alpha_4, \ldots, \alpha_8$ is even and the $\mathbb{Z}_2$-charge of α_3 is odd. It is easy to check that if we require the $\mathbb{Z}_2$-charge to be multiplicative ($q(\alpha+\beta) = q(\alpha)q(\beta)$), we get all $\mathbb{Z}_2$-charge assignments (3.1). The $\mathbb{Z}_2$-charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i\alpha_i$ ($0 \le k_i \in \mathbb{Z}$), is therefore $(-1)^{k_3}$.

Now, let us move on to the twisted sector of the orbifold. It consists of 16 M5-branes, which are associated with the imaginary root

$$
\tau_{M5} = (2,2,2,2,2,1,1,1,1,1)\,. \tag{3.2}
$$

We see that the $\mathbb{Z}_2$-charge associated with a generic root β is determined by $q(\beta) = (-1)^{(\beta|\tau_{\mathrm{M5}})}$.

3.2 $T^8/\mathbb{Z}_2$

Now we repeat the calculation for the orbifold $T^8/\mathbb{Z}_2$. This time the charge assignments are

$$
\begin{aligned}
g_{ij} &\to g_{ij}\,, && 1 \le i,j \le 2\,,\\
g_{iI} &\to -g_{iI}\,, && 1 \le i \le 2, \quad 3 \le I \le 10\\
g_{IJ} &\to g_{IJ}\,, && 3 \le I,J \le 10
\end{aligned}
$$

$$
\begin{aligned}
C_{12K} &\to -C_{12K}\,, \qquad 3 \le K \le 10\\
C_{iJK} &\to C_{iJK}\,, \qquad 1 \le i \le 2, \quad 3 \le J < K \le 10\\
C_{IJK} &\to -C_{IJK}\,, \qquad 3 \le I < J < K \le 10\,.
\end{aligned}
$$

Using (2.4), we find that the $\mathbb{Z}_2$-charge of $\alpha_{-1}, \alpha_1, \ldots, \alpha_7$ is even and the $\mathbb{Z}_2$-charge of α_0, α_8 is odd. The $\mathbb{Z}_2$-charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i \alpha_i$ $(0 \le k_i \in \mathbb{Z})$ is $(-1)^{k_0+k_8}$.

The twisted sector contains 16 M2-branes, which are associated with the imaginary root

$$
\tau_{M2} = (2,2,1,1,1,1,1,1,1,1)\,.
$$

We see that the charge of a generic root β is again determined by $(-1)^{(\beta|\tau_{\mathrm{M2}})}$.

3.3 $S^1/\mathbb{Z}_2$

We will now discuss the Hořava -Witten orbifold $S^1/\mathbb{Z}_2$. Let the S^1 be in the 10^{th} direction. According to [7], the $\mathbb{Z}_2$ acts on the bosonic fields of M-theory as follows

$$
\begin{aligned}
g_{ij} &\to g_{ij}\,, \qquad 1 \le i,j \le 9\,,\\
g_{i10} &\to -g_{i10}\,, \qquad 1 \le i \le 9\,,\\
C_{ijk} &\to -C_{ijk}\,, \qquad 1 \le i < j < k \le 9\,,\\
C_{ij10} &\to C_{ij10}\,, \qquad 1 \le i < j \le 9\,,
\end{aligned}
$$

The $\mathbb{Z}_2$ charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i \alpha_i$ $(0 \le k_i \in \mathbb{Z})$ is now defined as $(-1)^{k_7}$.

3.4 Type IA

If we switch the role of the 9^{th} and 10^{th} directions in the $S^1/\mathbb{Z}_2$ orbifold of section 3.3, and take the limit $M_p R_{10} \to 0$, we get type-IA [8]. The $\mathbb{Z}_2$ acts on the bosonic fields of M-theory as follows

$$
\begin{aligned}
g_{ij} &\to g_{ij}\,, \qquad i,j = 1\ldots 8, 10\,,\\
g_{i9} &\to -g_{i9}\,, \qquad i = 1\ldots 8, 10\,,\\
C_{ijk} &\to -C_{ijk}\,, \qquad i,j,k = 1\ldots 8, 10\,,\\
C_{ij9} &\to C_{ij9}\,, \qquad i,j = 1\ldots 8, 10\,.
\end{aligned}
$$

The $\mathbb{Z}_2$-charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i \alpha_i$ $(0 \le k_i \in \mathbb{Z})$ is now defined as $(-1)^{k_6+k_7}$.

Now let us analyze the twisted sector. The twisted sector of type-IA consists of 16 D8-branes [39, 8]. The D8-branes span directions $1\ldots 8$. If formally we lift them back to M-theory, they correspond to the imaginary root

$$
\tau_{D8} = (2,2,2,2,2,2,2,2,1,4)\,. \tag{3.3}
$$

The associated mass is [1]

$$
\frac{e^{\langle \tau_{D8}, \vec{h}\rangle}}{M_p^9 R_1 \cdots R_{10}} = M_p^{12} R_1 \cdots R_8 R_{10}^3 = \frac{1}{g_s} M_s^9 R_1 \cdots R_8\,,
$$

where $M_s = M_p^{3/2} R_{10}^{1/2}$ is the string scale, and $g_s = (M_p R_{10})^{3/2}$ is the string coupling constant.

JHEP06(2005)057

Note that for any root $\beta = \sum_{i=-1}^{8} k_i \alpha_i$, we have $(\beta|\tau_{D8}) = k_6 - 3k_7$, and therefore the $\mathbb{Z}_2$-charge of any root can be written as

$$q(\beta) = (-1)^{(\beta|\tau_{D8})}\,.$$

Of course, any other root of the form $\tau_{D8} + 2\alpha''$, with α'' in the root lattice, will have the same property. Had we chosen another direction, instead of the 10^{th}, on which to reduce to type-IA, say the 8^{th}, we would have gotten another imaginary root, say

$$\tau'_{D8} = (2,2,2,2,2,2,2,4,1,2)\,.$$

The mass associated with this root is

$$\frac{e^{\langle \tau'_{D8}, \vec{h}\rangle}}{M_p^9 R_1 \cdots R_{10}} = M_p^{12} R_1 \cdots R_7 R_8^3 R_{10}\,.$$

It is interesting to check what the mass of the object associated with τ'_{D8} is in string variables corresponding to the original reduction along the 10^{th} direction. We get [27]

$$M_p^{12} R_1 \cdots R_7 R_8^3 R_{10} = \frac{M_s^{11}}{g_s^3} R_1 \cdots R_7 R_8^3\,,$$

where we used

$$R_{10} = \frac{g_s}{M_s}\,, \qquad M_p = g_s^{-\frac{1}{3}} M_s\,.$$

Both τ_{D8} and τ'_{D8} determine the same $\mathbb{Z}_2$-charge

$$q(\beta) = (-1)^{(\beta|\tau_{D8})} = (-1)^{(\beta|\tau'_{D8})}\,.$$

So what distinguishes τ'_{D8}? The answer seems to be that the particular choice of simple roots (2.4) implies a particular asymptotic region (2.7) of $\vec{h}$-space, and the reduction to type-IA on the 8^{th} direction is less favorable than the reduction on the 10^{th}.

3.5 $T^9/\mathbb{Z}_2$

Now we repeat the calculation for the orbifold $T^9/\mathbb{Z}_2$. This time the charge assignments are

$$\begin{aligned} g_{1I} &\to -g_{1I}\,, \qquad 2 \leq I \leq 10 \\ g_{IJ} &\to g_{IJ}\,, \qquad 2 \leq I, J \leq 10 \\ C_{1JK} &\to -C_{1JK}\,, \qquad 2 \leq J < K \leq 10 \\ C_{IJK} &\to C_{IJK}\,, \qquad 2 \leq I < J < K \leq 10\,. \end{aligned}$$

Using (2.4), we find that the $\mathbb{Z}_2$-charge of $\alpha_0, \ldots, \alpha_8$ is even and the $\mathbb{Z}_2$-charge of α_{-1} is odd. The $\mathbb{Z}_2$ charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i \alpha_i$ $(0 \leq k_i \in \mathbb{Z})$ is $(-1)^{k_{-1}}$.

The twisted sector contains KK-particles, which are associated with the imaginary root

$$\tau_{KK} = (0,1,1,1,1,1,1,1,1,1)\,.$$

We see that the charge of a generic root β is again determined by $(-1)^{(\beta|\tau_{\mathrm{KK}})}$.

3.6 $T^4/\mathbb{Z}_2$

For the last example, we take the orbifold $T^4/\mathbb{Z}_2$. The charge assignments are

$$\begin{aligned}
g_{ij} &\to g_{ij}\,, \qquad 1 \le i, j \le 6\,,\\
g_{iI} &\to -g_{iI}\,, \qquad 1 \le i \le 6\,, \quad 7 \le I \le 10\\
g_{IJ} &\to g_{IJ}\,, \qquad 7 \le I, J \le 10\\
C_{ijk} &\to C_{ijk}\,, \qquad 1 \le i < j < k \le 6\,,\\
C_{ijK} &\to -C_{ijK}\,, \qquad 1 \le i < j \le 6\,, \quad 7 \le K \le 10\\
C_{iJK} &\to C_{iJK}\,, \qquad 1 \le i \le 6, \quad 7 \le J < K \le 10\\
C_{IJK} &\to -C_{IJK}\,, \qquad 7 \le I < J < K \le 10\,.
\end{aligned}$$

Using (2.4), we find that the $\mathbb{Z}_2$-charge of $\alpha_{-1}, \dots, \alpha_3, \alpha_5, \dots, \alpha_7$ is even and the $\mathbb{Z}_2$-charge of α_4, α_8 is odd. The $\mathbb{Z}_2$-charge of a generic root $\alpha = \sum_{i=-1}^{8} k_i\alpha_i$ $(0 \le k_i \in \mathbb{Z})$ is $(-1)^{k_4+k_8}$.

The twisted sectors contain KK-monopoles which are associated with the imaginary roots

$$\tau_{KKM} = (2,2,2,2,2,2,1,1,1,3)\,,$$

or any of the permutations

$$(2,2,2,2,2,2,1,1,3,1)\,, \quad (2,2,2,2,2,2,1,3,1,1)\,, \quad (2,2,2,2,2,2,3,1,1,1)\,.$$

(They are all equal mod 2.) We see that the charge of a generic root β is again determined by

$$(-1)^{(\beta|\tau_{KKM})}\,.$$

3.7 Summary

The branes of the twisted sector in a $\mathbb{Z}_2$ orbifold of T^{10} correspond to a positive isotropic imaginary root τ such that the $\mathbb{Z}_2$-charge of any root β is given by

$$q(\beta) = (-1)^{(\beta|\tau)}\,. \tag{3.4}$$

4. The untwisted sector and DE_{10}

We will now study in more detail the subalgebra of E_{10} that consists of all elements with even $\mathbb{Z}_2$-charge.

4.1 Definition of $\mathfrak{g}^{(\mathrm{inv})}$

For each of the $\mathbb{Z}_2$-orbifolds above, we define the "untwisted" sector $\mathfrak{g}^{(\mathrm{inv})}$ as the Lie subalgebra of E_{10} generated by all the E_{10} generators that correspond to roots with even charge. In particular, the Cartan subalgebra $\mathfrak{h}(E_{10}) \subset E_{10}$ is contained in $\mathfrak{g}^{(\mathrm{inv})}$, since its generators correspond to the root 0.

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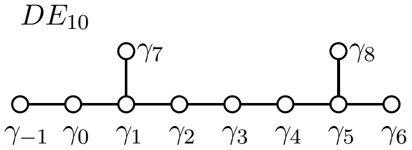

Figure 2: The Dynkin diagram of DE_{10}.

In section 3.7, we observed that the $\mathbb{Z}_2$-orbifold is associated with a positive prime isotropic imaginary root τ, and the $\mathbb{Z}_2$-even roots α are precisely those with even $(\alpha|\tau)$. We define the subset of even roots

$$\Delta^{(\mathrm{inv})} := \{\alpha \in \Delta\colon \quad (\alpha|\tau) \equiv 0 \mod 2\}\,, \qquad \Delta^{(\mathrm{inv})} \subset \Delta \tag{4.1}$$

and the sublattice

$$Q^{(\mathrm{inv})} := \{\alpha \in Q\colon \quad (\alpha|\tau) \equiv 0 \mod 2\}\,, \qquad Q^{(\mathrm{inv})} \subset Q\,. \tag{4.2}$$

We proceed to study the subalgebras $\mathfrak{g}^{(\mathrm{inv})}$. Since all positive prime isotropic imaginary roots τ are W-equivalent, all $\mathfrak{g}^{(\mathrm{inv})}$ algebras are isomorphic. It is therefore sufficient to concentrate on one particular orbifold, say $T^5 \times (T^5/\mathbb{Z}_2)$.

We will demonstrate that $\mathfrak{g}^{(\mathrm{inv})}$ contains a proper subalgebra that is isomorphic to DE_{10}. Here "proper" means that $\mathfrak{g}^{(\mathrm{inv})}$ is actually bigger than DE_{10}, as we will demonstrate in section 4.3. These observations have also been made, independently, in [21].

4.2 The Kac-Moody algebra DE_{10}

We begin with the salient features of the hyperbolic Kac-Moody algebra DE_{10}. The Dynkin diagram is given in figure 2.

We have the Cartan matrix:

$$A(DE_{10}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

Its inverse is

$$A(DE_{10})^{-1} = \begin{pmatrix} 0 & -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & -1 \\ -1 & -2 & -4 & -4 & -4 & -4 & -4 & -2 & -2 & -2 \\ -2 & -4 & -6 & -6 & -6 & -6 & -6 & -3 & -3 & -3 \\ -2 & -4 & -6 & -5 & -5 & -5 & -5 & -\frac{5}{2} & -3 & -\frac{5}{2} \\ -2 & -4 & -6 & -5 & -4 & -4 & -4 & -2 & -3 & -2 \\ -2 & -4 & -6 & -5 & -4 & -3 & -3 & -\frac{3}{2} & -3 & -\frac{3}{2} \\ -2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & -3 & -1 \\ -1 & -2 & -3 & -\frac{5}{2} & -2 & -\frac{3}{2} & -1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ -1 & -2 & -3 & -3 & -3 & -3 & -3 & -\frac{3}{2} & -1 & -\frac{3}{2} \\ -1 & -2 & -3 & -\frac{5}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & -\frac{3}{2} & 0 \end{pmatrix} . \tag{4.3}$$

The rows of the above matrix are the fundamental weights $\widetilde{\Lambda}_i$ ($i = -1, \ldots, 8$). They satisfy

$$(\widetilde{\Lambda}_i|\gamma_j) = \delta_{i,j}\,, \qquad i,j = -1, \ldots, 10\,. \tag{4.4}$$

The sum of the simple weights is (referring to figure 2),

$$\rho(DE_{10}) = \sum_{i=-1}^{8} \widetilde{\Lambda}_i = -14\gamma_{-1} - 29\gamma_0 - 45\gamma_1 - 40\gamma_2 - 36\gamma_3 - 33\gamma_4 - 31\gamma_5 - 15\gamma_6 - 22\gamma_7 - 15\gamma_8\,. \tag{4.5}$$

We denote the root space decomposition of DE_{10} by

$$DE_{10} = \bigoplus_{\alpha\in\Delta(DE_{10})} \mathfrak{g}(DE_{10})_\alpha\,, \quad \mathfrak{g}(DE_{10})_\alpha := \{x \in DE_{10}\colon \quad [h,x] = \langle\alpha, h\rangle x \quad \forall h \in \mathfrak{h}(DE_{10})\}\,. \tag{4.6}$$

We will also need the notation of the dual basis $\gamma^\vee_{-1}, \ldots, \gamma^\vee_8$, which is a basis of the Cartan subalgebra $\mathfrak{h}(DE_{10})$ such that

$$\langle\gamma^\vee_i, \gamma_j\rangle = A(DE_{10})_{ij}\,, \qquad i,j = -1, \ldots, 8\,.$$

We denote by $P(DE_{10}) \subset \mathfrak{h}(DE_{10})^*$ the weight lattice

$$P(DE_{10}) := \{\widetilde{\Lambda} \in \mathfrak{h}(DE_{10})^*\colon \quad (\widetilde{\Lambda}|\alpha) \in \mathbb{Z} \quad \forall\alpha \in Q(DE_{10})\} \equiv \sum_{i=-1}^{8} \mathbb{Z}\widetilde{\Lambda}_i. \tag{4.7}$$

4.3 The subgroup $DE_{10} \subset \mathfrak{g}^{(\mathrm{inv})}$

We will now show that $\mathfrak{g}^{(\mathrm{inv})}$ contains a subgroup DE_{10}. For concreteness, we will refer to the $T^5/\mathbb{Z}_2$ orbifold (discussed in section 3.1). First, we look for a basis of $Q^{(\mathrm{inv})}$ (defined in (4.2)) that is comprised of real roots. The $\mathbb{Z}_2$-charge is determined by k_3 (mod 2), so all simple roots $\alpha_{-1}, \ldots, \alpha_8$ except α_3 belong to $Q^{(\mathrm{inv})}$. Let us find another real root of E_{10} that completes these 9 simple roots to a basis of $Q^{(\mathrm{inv})}$. We need a root with an even coefficient of k_3, and it is not hard to check that

$$\xi = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8 \to (0,0,0,1,1,0,0,0,0,1)$$

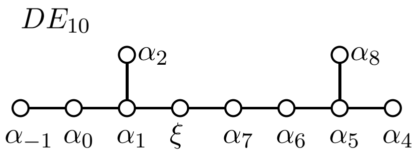

Figure 3: The Dynkin diagram of the $\mathbb{Z}_2$-invariant subalgebra $DE_{10} \subset E_{10}$ that is associated with the $T^5/\mathbb{Z}_2$ orbifold.

is the minimal root with this property. Furthermore,

$$Q^{(\mathrm{inv})} = \mathbb{Z}\alpha_{-1} + \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_4 + \mathbb{Z}\alpha_5 + \mathbb{Z}\alpha_6 + \mathbb{Z}\alpha_7 + \mathbb{Z}\alpha_8 + \mathbb{Z}\xi \,,$$

and

$$\Delta^{(\mathrm{inv})} = Q^{(\mathrm{inv})} \cap \Delta \,.$$

The inner products among the real roots α_{-1}, α_0, α_1, α_2, α_4, α_5, α_6, α_7, α_8 and ξ are depicted in a Dynkin-like diagram in figure 3.

Comparing to figure 2, we can define an isometry $\upsilon : \mathfrak{h}(DE_{10})^* \to \mathfrak{h}(E_{10})^*$ by

$$\begin{aligned} &\upsilon(\gamma_{-1}) = \alpha_{-1}\,, && \upsilon(\gamma_0) = \alpha_0\,, && \upsilon(\gamma_1) = \alpha_1\,, && \upsilon(\gamma_2) = \xi\,, && \upsilon(\gamma_3) = \alpha_7\,, \\ &\upsilon(\gamma_4) = \alpha_6\,, && \upsilon(\gamma_5) = \alpha_5\,, && \upsilon(\gamma_6) = \alpha_4\,, && \upsilon(\gamma_7) = \alpha_2\,, && \upsilon(\gamma_8) = \alpha_8\,, \end{aligned} \tag{4.8}$$

Using the nondegenerate bilinear forms $(\cdot|\cdot)$ on E_{10} and DE_{10} we can identify $\mathfrak{h}(DE_{10})^*$ with $\mathfrak{h}(DE_{10})$ and $\mathfrak{h}(E_{10})^*$ with $\mathfrak{h}(E_{10})$. The map υ then defines an isomorphism between the Cartan subalgebras of E_{10} and DE_{10}. We can extend this map to an injective Lie algebra homomorphism

$$\tilde{\upsilon} : DE_{10} \to \mathfrak{g}^{(\mathrm{inv})} \subset E_{10}\,. \tag{4.9}$$

The details can be found in appendix A.

4.4 Comparing DE_{10} to $\mathfrak{g}^{(\mathrm{inv})}$

We have seen in section 4.3 that DE_{10} is a subalgebra of $\mathfrak{g}^{(\mathrm{inv})}$. We will now demonstrate that it is a *proper* subalgebra, i.e. $\mathfrak{g}^{(\mathrm{inv})}$ is bigger. We note that DE_{10} has an affine $\widehat{D}_8$ subalgebra, and we will study the algebras DE_{10} and $\mathfrak{g}^{(\mathrm{inv})}$ as representations of $\widehat{D}_8$. (Similar observations have been made independently in [21], but based on a D_9 rather than $\widehat{D}_8$ subalgebra.)

Both DE_{10} and $\mathfrak{g}^{(\mathrm{inv})}$ decompose into irreducible representations of $\widehat{D}_8$. The complete decomposition is unknown. It is, however, possible to calculate the number of times that any representation of $\widehat{D}_8$ with affine-level $k = 1$ or 2 appears in DE_{10} and in E_{10}.

Recall that affine $\widehat{D}_8 \simeq \widehat{\mathfrak{so}}(16)$ is the Lie algebra with generators J^a_n, K, L_0 [where $n \in \mathbb{Z}$ and $a = 1, \ldots, 120 = \dim \mathfrak{so}(16)$] and commutation relations [40]

$$[J^a_n, J^b_m] = f^{ab}{}_c J^c_{m+n} + n\delta_{m,-n}\delta^{ab}K\,, \qquad [K, J^a_n] = [K, L_0] = 0\,, \qquad [L_0, J^a_n] = -nJ^a_n\,, \tag{4.10}$$

where $f^{ab}{}_c$ are the structure constants of $\mathfrak{so}(16)$ in a basis where the Killing form is δ^{ab}.

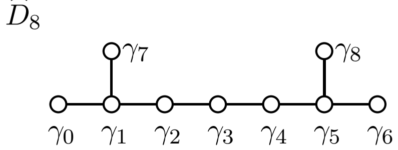

Figure 4: The Dynkin diagram of the $\widehat{D}_8$ subalgebra.

Looking at the Dynkin diagram of DE_{10} (figure 2), we see that if we drop the node γ_{-1}, we get the Dynkin diagram of $\widehat{D}_8$ (figure 4). Define the DE_{10}-root

$$\tilde{\delta}:=\gamma_0+2\gamma_1+2\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6+\gamma_7+\gamma_8\,.$$

Then the $\widehat{D}_8\subset DE_{10}$ subalgebra can be defined as the sum of the root spaces $\mathfrak{g}(DE_{10})_\alpha$ that correspond to roots of the form $\alpha=\sum_0^8 k_i\gamma_i$. This can also be written as,

$$\widehat{D}_8\simeq\bigoplus_{(\alpha|\tilde{\delta})=0}\mathfrak{g}(DE_{10})_\alpha\,.$$

In particular $\mathfrak{h}(DE_{10})\subset\widehat{D}_8$. The elements $\gamma_{-1}^\vee,\ldots,\gamma_8^\vee$, which form a basis for $\mathfrak{h}(DE_{10})$, are identified with the following generators of $\widehat{D}_8$ [25]: $\gamma_1^\vee,\ldots,\gamma_8^\vee$ are identified with appropriate linear combinations of the J_0^a's that span the Cartan subalgebra of D_8; the element

$$\tilde{\delta}^\vee:=\gamma_0^\vee+2\gamma_1^\vee+2\gamma_2^\vee+2\gamma_3^\vee+2\gamma_4^\vee+2\gamma_5^\vee+\gamma_6^\vee+\gamma_7^\vee+\gamma_8^\vee\,,$$

is identified with K, and $\gamma_{-1}^\vee$ is identified with L_0.

Furthermore, for any positive integer k, the subspace of all level-k root-spaces,

$$(DE_{10})_{[k]}:=\bigoplus_{(\alpha|\tilde{\delta})=k}\mathfrak{g}(DE_{10})_\alpha\,,\tag{4.11}$$

is a representation of $\widehat{D}_8$ of affine-level k (i.e. the central element K of $\widehat{D}_8$ is constant on $(DE_{10})_{[k]}$ with value k). The weights of this representation are bounded from above, since only negative roots α appear in (4.11). We need the decomposition of $(DE_{10})_{[k]}$ into irreducible representations of $\widehat{D}_8$. There is a standard method to find this decomposition [41]–[43] that is also nicely explained in [44].

First, recall that an irreducible highest-weight representation of $\widehat{D}_8\simeq\widehat{\mathfrak{so}}(16)$ is defined by the highest-weight $\widetilde{\Lambda}$. This weight $\widetilde{\Lambda}$ is a 10-dimensional vector and can naturally be identified with an element of $\mathfrak{h}(DE_{10})^*$, since $\mathfrak{h}(DE_{10})=\mathfrak{h}(\widehat{D}_8)$. $\widetilde{\Lambda}$ is a linear combination of the fundamental weights $\widetilde{\Lambda}_{-1},\ldots,\widetilde{\Lambda}_8$ (4.4). This representation is denoted by $L(\widetilde{\Lambda})$. It turns out ([25, section 12]) that

$$(DE_{10})_{[1]}\simeq L(-\gamma_{-1})\equiv L(\widetilde{\Lambda}_0+2\tilde{\delta}).\tag{4.12}$$

Here the representation $L(\widetilde{\Lambda}_0+2\tilde{\delta})$ can be realized as the Hilbert space of 16 chiral Majorana fermions in 1+1D with anti-periodic boundary conditions [40].

The decomposition of $(DE_{10})_{[2]}$ into irreducible $\widehat{D}_8$ representations is a bit more complicated. It can be shown that $(DE_{10})_{[2]}$ can be represented as a certain coset of the anti-symmetric tensor product $\wedge^2(DE_{10})_{[1]} = \wedge^2 L(\widetilde{\Lambda}_0 + 2\tilde{\delta})$. Specifically,

$$(DE_{10})_{[2]} = \frac{\wedge^2 L(\widetilde{\Lambda}_0 + 2\tilde{\delta})}{L(\widetilde{\Lambda}_1 + 3\tilde{\delta})} \,.$$

The tensor product $\wedge^2 L(\widetilde{\Lambda}_0 + 2\tilde{\delta})$ can be decomposed into irreducible $\widehat{D}_8$ representations using the *coset construction.* The result is

$$(DE_{10})_{[2]} \simeq \bigoplus_{n=0}^{\infty} \tilde{C}_n^{(1)} L(\widetilde{\Lambda}_1 + (3-n)\tilde{\delta}) \oplus \bigoplus_{n=0}^{\infty} \tilde{C}_n^{(5)} L(\widetilde{\Lambda}_5 + (1-n)\tilde{\delta}) \tag{4.13}$$

where the integer multiplicities $\tilde{C}_n^{(1)}$ and $\tilde{C}_n^{(5)}$ are given by the generating functions

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{C}_n^{(1)} q^n &= \frac{1}{\varphi(q)} - 1 = q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots \,, \\
\sum_{n=0}^{\infty} \tilde{C}_n^{(5)} q^n &= \frac{1}{\varphi(q)} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots \,,
\end{aligned}$$

where

$$\varphi(q) := \prod_{m=1}^{\infty} (1-q^m) = \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{1}{2}n(3n-1)} + q^{\frac{1}{2}n(3n+1)}\right) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots \,,$$

The generating function $1/\varphi(q)$ is the character of a Virasoro representation with central charge $c = 1$ (a free boson). It appears in this context because the coset construction $\widehat{\mathfrak{so}}(2n)_1 \otimes \widehat{\mathfrak{so}}(2n)_1 / \widehat{\mathfrak{so}}(2n)_2$ has central charge $c = 1$. We have also used a theorem from [42] which states that for an affine Lie algebra of type ADE with "odd exponents" (see [25, see section 14.1] for a definition), if a highest-weight representation $L(M)$ appears in $\wedge^2 L(\Lambda)$, for $\Lambda \in P^+$ of level-1, then $\mathrm{ht}(2\Lambda - M)$ must be odd (where ht is the "height" of the root, i.e. the sum of the coefficients in the root's decomposition into simple roots). Note that this theorem applies to both E_9 and $\widehat{D}_8$. For more details on the relevant $\widehat{\mathfrak{so}}(2n)_1 \otimes \widehat{\mathfrak{so}}(2n)_1 / \widehat{\mathfrak{so}}(2n)_2$ coset construction, see [40, 42], [45]–[48].

To compare DE_{10} to $\mathfrak{g}^{(\mathrm{inv})} \subset E_{10}$, we first define

$$\delta := \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \in \Delta(E_{10}) \,, \tag{4.14}$$

and

$$(E_{10})_{[k]} := \bigoplus_{(\alpha|\delta)=k} \mathfrak{g}(E_{10})_\alpha \,. \tag{4.15}$$

We then define

$$(\mathfrak{g}^{(\mathrm{inv})})_{[k]} := \mathfrak{g}^{(\mathrm{inv})} \cap (E_{10})_{[k]} \,.$$

Since $\upsilon(\gamma_{-1}) = \alpha_{-1}$ (see (4.8)) and $\upsilon(\tilde{\delta}) = \delta$, every level-$k$ root of DE_{10} (with respect to $\tilde{\delta}$) is also a level-k root of E_{10} with respect to δ. We therefore have

$$\tilde{\upsilon}((DE_{10})_{[k]}) \subseteq (\mathfrak{g}^{(\mathrm{inv})})_{[k]} \,,$$

where $\tilde{\upsilon}$ was defined in (4.9).

JHEP06(2005)057

E_{10} has an E_9 (affine $\widehat{E}_8$) subalgebra that corresponds to the nodes $\alpha_0, \ldots, \alpha_8$ in figure 1, and $(E_{10})_{[k]}$ is an E_9-representation. $(\mathfrak{g}^{(\mathrm{inv})})_{[k]}$, on the other hand, is not an E_9-representation, since E_9 contains $\mathbb{Z}_2$-odd generators. However, the direct sum of the $\mathbb{Z}_2$-even root spaces of E_9 is isomorphic to $\widehat{D}_8$. Thus, $(\mathfrak{g}^{(\mathrm{inv})})_{[k]}$ is a $\widehat{D}_8$-representation. In order to complete the comparison of DE_{10} to $\mathfrak{g}^{(\mathrm{inv})}$ at low levels, we need to know how $(\mathfrak{g}^{(\mathrm{inv})})_{[k]}$ decomposes into irreducible representations of $\widehat{D}_8$. We can do this by decomposing $(E_{10})_{[k]}$ into irreducible representations of E_9, then decomposing these representations under $\tilde{\upsilon}(\widehat{D}_8) \subset E_9$, and then intersecting with $\mathfrak{g}^{(\mathrm{inv})}$.

The embedding $\widehat{D}_8 \simeq \tilde{\upsilon}(\widehat{D}_8) \subset E_9$ is easy to describe. It is the extension of the minimal embedding $\mathfrak{so}(16) \subset E_8$ (under which the adjoint representation **248** of E_8 decomposes as the sum of the adjoint **120** and one of the spinor **128** representations of $\mathfrak{so}(16)$) to affine algebras.

The decomposition of $(E_{10})_{[k]}$ at level $k = 1, 2$ was calculated in [41]–[43] (and was extended to $k = 3$ in [44], but we will not use this result here). It is given by

$$(E_{10})_{[1]} \simeq L(-\alpha_{-1}) \equiv L(\Lambda_0 + 2\delta)\,, \tag{4.16}$$

and

$$(E_{10})_{[2]} \simeq \bigoplus_{n=0}^{\infty} c_n L(\Lambda_1 + (3-n)\delta) \tag{4.17}$$

with the generating function

$$\sum_{n=0}^{\infty} c_n q^n = \prod_{n=1}^{\infty}(1+q^n) - 1 = \frac{\varphi(q^2)}{\varphi(q)} - 1\,.$$

To compare with (4.12) and (4.13), we need to decompose (4.16) and (4.17) under $\widehat{D}_8 \subset E_9$. For level-1 we have

$$L(\Lambda_0 + 2\delta) \simeq L(\widetilde{\Lambda}_1 + 2\tilde{\delta}) \oplus L(\widetilde{\Lambda}_6 + 2\tilde{\delta})\,. \tag{4.18}$$

Note that on the left-hand side we have an irreducible representation of E_9, and on the right-hand side we have irreducible representations of $\widehat{D}_8$. To complete the calculation of $(\mathfrak{g}^{(\mathrm{inv})})_{[1]}$ we must intersect (4.18) with $\mathfrak{g}^{(\mathrm{inv})}$.

From (4.3) we find

$$\begin{aligned}
-\widetilde{\Lambda}_1 &= 2\gamma_{-1} + 4\gamma_0 + 6\gamma_1 + 6\gamma_2 + 6\gamma_3 + 6\gamma_4 + 6\gamma_5 + 3\gamma_6 + 3\gamma_7 + 3\gamma_8\,,\\
-\widetilde{\Lambda}_5 &= 2\gamma_{-1} + 4\gamma_0 + 6\gamma_1 + 5\gamma_2 + 4\gamma_3 + 3\gamma_4 + 2\gamma_5 + \gamma_6 + 3\gamma_7 + \gamma_8\,,\\
-\widetilde{\Lambda}_6 &= \gamma_{-1} + 2\gamma_0 + 3\gamma_1 + \frac{5}{2}\gamma_2 + 2\gamma_3 + \frac{3}{2}\gamma_4 + \gamma_5 + \frac{3}{2}\gamma_7 + \frac{1}{2}\gamma_8\,.
\end{aligned}$$

(We have also listed $\widetilde{\Lambda}_5$, which will be needed later on.) Since

$$-\upsilon(\widetilde{\Lambda}_6) = \alpha_{-1} + 2\alpha_0 + 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

is a $\mathbb{Z}_2$-odd level-1 root, all the weights of the module $L(\widetilde{\Lambda}_6 + 2\tilde{\delta})$ map, under υ, to $\mathbb{Z}_2$-odd roots of E_{10}. the intersection $\mathfrak{g}^{(\mathrm{inv})} \cap L(\Lambda_0 + 2\delta)$ therefore leaves only $L(\widetilde{\Lambda}_1 + 2\tilde{\delta})$, and thus

$$(DE_{10})_{[1]} = (\mathfrak{g}^{(\mathrm{inv})})_{[1]}\,.$$

JHEP06(2005)057

There is another way to arrive at this result. Recall that $L(\Lambda_0+2\delta)$ can be realized as the Hilbert space of 16 1+1D chiral Majorana fermions with either all periodic or all anti-periodic boundary conditions [49]. According to [40], 16 Majorana fermions also realize $\widehat{\mathfrak{so}}(16)$. The representation $L(\widetilde{\Lambda}_1+2\tilde{\delta})$ corresponds to the sector with anti-periodic boundary conditions and odd fermion number, and the representation $L(\widetilde{\Lambda}_6+2\tilde{\delta})$ corresponds to the sector with periodic boundary conditions. It is not hard to check that the $\mathbb{Z}_2$-charge is $+$ or $-$ according to whether the boundary conditions are anti-periodic or periodic. Thus,

$$(\mathfrak{g}^{(\mathrm{inv})})_{[1]} \simeq L(\widetilde{\Lambda}_1+2\tilde{\delta}) \simeq (DE_{10})_{[1]} \,.$$

At level-2 such an equality will no longer hold.

The decomposition under $\widehat{D}_8$ of the $\mathbb{Z}_2$-invariant part of the level-2 E_9 representations that appear in (4.17) is given by

$$L(\Lambda_1+3\delta)^{(inv)} \simeq \bigoplus_{m=0}^{\infty} C_m^{(1)\prime} L(\widetilde{\Lambda}_1+(3-m)\tilde{\delta}) \oplus \bigoplus_{m=0}^{\infty} C_m^{(5)\prime} L(\widetilde{\Lambda}_5+(3-m)\tilde{\delta}) \tag{4.19}$$

with

$$\sum_{n=0}^{\infty} C_n^{(1)\prime} q^n = \prod_{n=1}^{\infty}(1+q^n) = \frac{\varphi(q^2)}{\varphi(q)}, \qquad \sum_{n=0}^{\infty} C_n^{(5)\prime} q^n = q\prod_{n=1}^{\infty}(1+q^n) = \frac{q\varphi(q^2)}{\varphi(q)} \,.$$

These expressions can be derived from the coset construction $(E_9)_{k=2}/(\widehat{D}_8)_{k=2}$ which has central charge $c=\frac{1}{2}$.

Combining (4.17) with (4.19) we obtain

$$(\mathfrak{g}^{(\mathrm{inv})})_{[2]} \simeq \bigoplus_{n=0}^{\infty} C_n^{(1)} L(\widetilde{\Lambda}_1+(3-n)\tilde{\delta}) \oplus \bigoplus_{n=0}^{\infty} C_n^{(5)} L(\widetilde{\Lambda}_5+(3-n)\tilde{\delta}) \tag{4.20}$$

with

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(1)} q^n &= \prod_{n=1}^{\infty}(1+q^n)^2 - \prod_{n=1}^{\infty}(1+q^n) \\
&= q+2q^2+4q^3+7q^4+11q^5+18q^6+27q^7+\cdots \\
\sum_{n=0}^{\infty} C_n^{(5)} q^n &= q\Big[\prod_{n=1}^{\infty}(1+q^n)^2 - \prod_{n=1}^{\infty}(1+q^n)\Big] \\
&= q^2+2q^3+4q^4+7q^5+11q^6+18q^7+27q^8+\cdots \,.
\end{aligned}$$

Now we can compare (4.20) to (4.13). We have

$$(\mathfrak{g}^{(\mathrm{inv})})_{[2]}/(DE_{10})_{[2]} \simeq \bigoplus_{n=0}^{\infty} \Delta C_n^{(1)} L(\widetilde{\Lambda}_1+(3-n)\tilde{\delta}) \oplus \bigoplus_{n=0}^{\infty} \Delta C_n^{(5)} L(\widetilde{\Lambda}_5+(2-n)\tilde{\delta})$$

with

$$\sum_{n=0}^{\infty} \Delta C_n^{(1)} q^n = \prod_{n=1}^{\infty}(1+q^n)^2 - \prod_{n=1}^{\infty}(1+q^n) - \prod_{n=1}^{\infty}(1-q^n)^{-1} + 1$$

$$
\begin{aligned}
&= q^3 + 2q^4 + 4q^5 + 7q^6 + 12q^7 + \cdots \\
\sum_{n=0}^{\infty} \Delta C_n^{(5)} q^n &= \prod_{n=1}^{\infty}(1+q^n)^2 - \prod_{n=1}^{\infty}(1+q^n) - q\prod_{n=1}^{\infty}(1-q^n)^{-1} \\
&= q^2 + 2q^3 + 4q^4 + 6q^5 + 11q^6 + 16q^7 + 25q^8 + \cdots
\end{aligned}
$$

We see that $\mathfrak{g}^{(\mathrm{inv})}$ is strictly bigger than DE_{10} at level 2. For example, the DE_{10} and $\mathfrak{g}^{(\mathrm{inv})}$ multiplicities of the E_{10} root

$$
\begin{aligned}
-\upsilon(\widetilde{\Lambda}_1) &= 2\alpha_{-1} + 4\alpha_0 + 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 15\alpha_4 + 18\alpha_5 + 12\alpha_6 + 6\alpha_7 + 9\alpha_8 \\
&\to (2,2,2,3,3,3,3,3,3,3)
\end{aligned}
$$

are 711 and 727, respectively. To calculate these multiplicities, one needs to know the multiplicities of $\widetilde{\Lambda}_1$ in $L(\widetilde{\Lambda}_1 + (3-n)\tilde{\delta})$ for $n = 1,2,3$, and also the multiplicities of $\widetilde{\Lambda}_1$ in $L(\widetilde{\Lambda}_5 + (2-n)\tilde{\delta})$ for $n = 0,1,2$. These can be found for example in [50] and are given by

$$
\begin{aligned}
&\sum \mathrm{mult}(\widetilde{\Lambda}_1 + (3-n)\tilde{\delta} \text{ in } L(\widetilde{\Lambda}_1 + 3\tilde{\delta}))q^n = \\
&\qquad = 1 + 27q + 362q^2 + 3377q^3 + 24831q^4 + 153635q^5 + \cdots \\
&\sum \mathrm{mult}(\widetilde{\Lambda}_1 + (3-n)\tilde{\delta} \text{ in } L(\widetilde{\Lambda}_5 + \tilde{\delta}))q^n = \\
&\qquad = 15q^2 + 277q^3 + 2917q^4 + 22740q^5 + \cdots
\end{aligned}
$$

At the next order, the DE_{10} and $\mathfrak{g}^{(\mathrm{inv})}$ multiplicities of the E_{10} root

$$
\begin{aligned}
\upsilon(\delta - \widetilde{\Lambda}_1) &= 2\alpha_{-1} + 5\alpha_0 + 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 20\alpha_4 + 24\alpha_5 + 16\alpha_6 + 8\alpha_7 + 12\alpha_8 \\
&\to (2,3,3,4,4,4,4,4,4,4)
\end{aligned}
$$

are 7411 and 7747, respectively.

5. The Kac-Moody algebra $DE_{18(10)}$

In section 4 we studied the "untwisted sectors" of the E_{10} $\mathbb{Z}_2$-orbifolds. We defined $\mathfrak{g}^{(\mathrm{inv})}$ and discovered that the hyperbolic Kac-Moody algebra DE_{10} is contained in it. On physical grounds, we know that there should also be "twisted sectors" associated with the 16 additional objects (exceptional branes) that are present in the M-theory compactification on $T^{10}/\mathbb{Z}_2$. In this section we will see that both twisted and untwisted sectors are unified inside a bigger structure — a coset of the Kac-Moody algebra $DE_{18(10)}$.

This algebra, that will be defined in section 5.3 below, contains a natural $\mathfrak{so}(16)$ subalgebra. We will see that this number 16 is directly related to the existence of 16 exceptional branes. We will propose that the decomposition of $DE_{18(10)}$ into irreducible representations of $\mathfrak{so}(16)$ has a physical meaning in terms of the $\mathbb{Z}_2$-orbifold of M-theory on T^{10} as follows. The entire "untwisted" sector is related to elements of $DE_{18(10)}$ that are in singlet representations **1** of $\mathfrak{so}(16)$ (i.e. commute with $\mathfrak{so}(16)$). Physical modes that are related to a single exceptional brane correspond to elements of $DE_{18(10)}$ that fall into the fundamental representation **16** of $\mathfrak{so}(16)$. Physical modes that are related to pairs of exceptional branes correspond to elements of $DE_{18(10)}$ that fall into either the anti-symmetric tensor representation **120** or the traceless symmetric tensor representation **135**, and so on.

5.1 Basic Properties of DE_{18}

The $\mathbb{Z}_2$-orbifolds that we are studying are dual to heterotic string theory compactified on T^9. As argued in [51], $\mathfrak{so}(24,8,\mathbb{R})$ is the relevant Lie algebra for heterotic string theory on T^7. The Dynkin diagram of the complexified Lie algebra is D_{16}. The extension to heterotic string theory on T^8 is the affine $\widehat{\mathfrak{so}}(24,8,\mathbb{R})$ ([17]–[19] and references therein). The extension to heterotic string theory on T^9 is naturally achieved by adding one more node. There are, of course, many ways to add a node to a given Dynkin diagram, but it turns out that the right one is the Dynkin diagram DE_{18}. The relation between these three Dynkin diagrams is depicted in figure 5.

5.2 The Cartan matrix

The Cartan matrix for DE_{18} with respect to the basis $\beta_{-1},\ldots,\beta_{16}$ is given by

$$A_{18} = \begin{pmatrix}
2 & -1 & & & & & & & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & -1 & & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & & & & & & & & \\
 & & & -1 & 2 & -1 & & & & & & & & & & & & \\
 & & & & -1 & 2 & -1 & & & & & & & & & & & \\
 & & & & & -1 & 2 & -1 & & & & & & & & & & \\
 & & & & & & -1 & 2 & -1 & & & & & & & & & \\
 & & & & & & & -1 & 2 & & -1 & & & & & & & \\
 & & -1 & & & & & & & 2 & & & & & & & & \\
 & & & & & & & & -1 & & 2 & -1 & & & & & & \\
 & & & & & & & & & & -1 & 2 & -1 & & & & & \\
 & & & & & & & & & & & -1 & 2 & -1 & & & & \\
 & & & & & & & & & & & & -1 & 2 & -1 & & & \\
 & & & & & & & & & & & & & -1 & 2 & -1 & & \\
 & & & & & & & & & & & & & & -1 & 2 & -1 & -1 \\
 & & & & & & & & & & & & & & & -1 & 2 & \\
 & & & & & & & & & & & & & & & -1 & & 2
\end{pmatrix}. \qquad (5.1)$$

The determinant is

$$\det A_{18} = -4,$$

and the inverse Cartan matrix A_{18}^{-1} is given by,

$$
\begin{pmatrix}
0 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 \\
-1 & -2 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -2 & -4 & -4 & -4 & -4 & -4 & -4 & -2 & -2 \\
-2 & -4 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -3 & -6 & -6 & -6 & -6 & -6 & -6 & -3 & -3 \\
-2 & -4 & -6 & -5 & -5 & -5 & -5 & -5 & -5 & -3 & -5 & -5 & -5 & -5 & -5 & -5 & -\frac{5}{2} & -\frac{5}{2} \\
-2 & -4 & -6 & -5 & -4 & -4 & -4 & -4 & -4 & -3 & -4 & -4 & -4 & -4 & -4 & -4 & -2 & -2 \\
-2 & -4 & -6 & -5 & -4 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -\frac{3}{2} & -\frac{3}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & -1 & -3 & -1 & -1 & -1 & -1 & -1 & -1 & -\frac{1}{2} & -\frac{1}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -2 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -1 & -3 & -3 & -3 & -3 & -3 & -3 & -\frac{3}{2} & -\frac{3}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 2 & 3 & 3 & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 2 & 3 & 4 & 4 & 4 & 2 & 2 \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 2 & 3 & 4 & 5 & 5 & \frac{5}{2} & \frac{5}{2} \\
-2 & -4 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & -3 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 3 \\
-1 & -2 & -3 & -\frac{5}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & -\frac{3}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & 2 & \frac{3}{2} \\
-1 & -2 & -3 & -\frac{5}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & -\frac{3}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{3}{2} & 2
\end{pmatrix} .
$$

The rows of A_{18}^{-1} will be useful to us as follows. Define the *fundamental weights* $\tilde{\Lambda}'_i \in \mathfrak{h}^*(DE_{18})$ $(i = -1, \dots 16)$ to satsify

$$\langle \tilde{\Lambda}'_i, \beta_j \rangle = \delta_{ij}, \qquad i, j = -1, \dots, 16.$$

Then $\tilde{\Lambda}'_i$ can be explicitly written as a linear combination of the simple roots β_j $(j = -1, \dots, 16)$ with coefficents given by the i^{th} row of $(A_{18})^{-1}$.

5.3 The real form $DE_{18(10)}$

We need a certain *real form* of the complexified Lie algebra DE_{18}. The real form is defined by the Tits-Satake diagram — a choice of "black" and "white" nodes on the Dynkin diagram, as depicted in figure 6.

It corresponds to a choice of anti-linear involution on the complexified Lie algebra DE_{18}, given by its action on the Chevalley generators that can be constructed as follows [52] (see also [53] for a comprehensive discussion).

First, we construct a linear involution ω^* on $\mathfrak{h}(DE_{18})^*$ that preserves the Killing form $(\cdot|\cdot)$. It is defined by its action on the simple roots

$$
\begin{aligned}
\omega^*(\beta_i) &= \beta_i \quad \text{for } i = -1, \dots 6 \text{ and } i = 8 \\
\omega^*(\beta_7) &= \beta_7 + 2\left(\sum_{i=1}^{6} \beta_{8+i}\right) + \beta_{15} + \beta_{16}\,, \\
\omega^*(\beta_i) &= -\beta_i \quad \text{for } i = 9, \dots 16\,.
\end{aligned}
$$

In general, ω^* is determined by the requirement that:

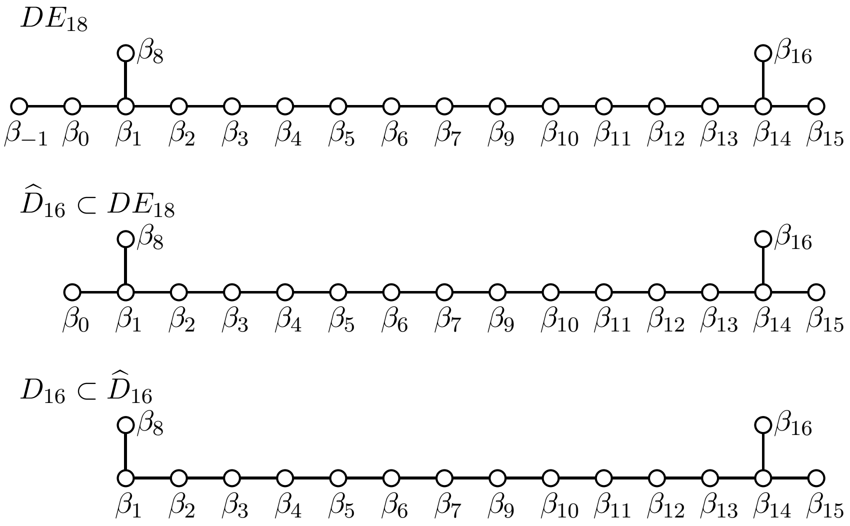

Figure 5: The Dynkin diagram of DE_{18} and its subdiagrams $\hat{D}_{16}$ and D_{16}.

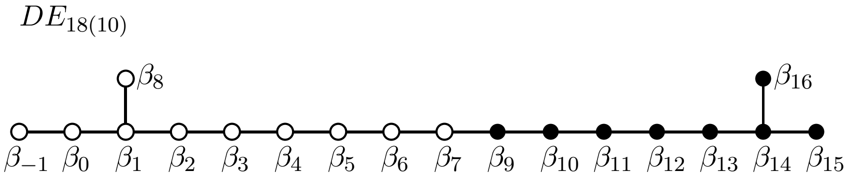

Figure 6: The Dynkin diagram of the real form $DE_{18(10)}$ of DE_{18}. The black and white nodes determine the real form.

- ω^* preserves the bilinear form $(\cdot|\cdot)$;
- $\omega^*(\beta_i) = -\beta_i$ for every "black" simple root β_i;
- For every "white" simple root β_j, we require $\omega^*(\beta_j) - \beta_j$ to be a linear combination of the "black" simple roots only.

Now that we have ω^* we can construct an anti-linear involution ω on the Lie algebra DE_{18} itself. We pick a set of Chevalley generators h_i, e_i, f_i $(i = -1, \ldots, 16)$ such that $h_i \in \mathfrak{h}(DE_{18})$, e_i generates the root space of α_i, and f_i generates the root space of $-\alpha_i$, and [25],

$$[e_i, f_i] = h_i\,, \qquad [h_i, e_i] = 2e_i\,, \qquad [h_i, f_i] = -2f_i\,.$$

Now note that $\omega^*(\beta_i)$ is a real root for $i = -1, \ldots, 16$. Its root space is therefore 1-dimensional, and a generator of this root space is uniquely determined, up to a multiplicative constant. We can now define the *anti-linear* involution ω in such a way that

$$\omega(e_i) \in \mathfrak{g}(DE_{18})_{\omega^*(\beta_i)}\,.$$

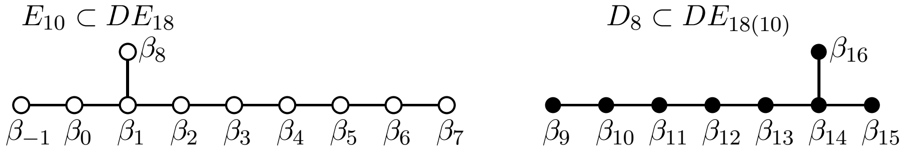

Figure 7: We split the Dynkin diagram of figure 6 into two sub-diagrams. The Dynkin sub-diagrams are E_{10} and D_8.

In particular, we can take $\omega(e_i) = -f_i$ and $\omega(f_i) = -e_i$ for $i = 9, \dots, 16$. For $i = -1, \dots, 6, 8$ we can take $\omega(e_i) = e_i$ and $\omega(f_i) = f_i$, and $\omega(e_7)$ has to be chosen as a generator of the root space of $\omega^*(\beta_7)$. This determines the action of ω on $\mathfrak{h}(DE_{18})$, and the requirement that $[\omega(e_7), \omega(f_7)] = \omega(h_7)$ then determines the normalization of $\omega(f_7)$. The real Lie algebra $DE_{18(10)}$ is then defined as the invariant subalgebra

$$DE_{18(10)} := \{x \in DE_{18}: \quad \omega(x) = x\}\,.$$

Note that

$$h_{-1}, \dots, h_6, h_8, ih_9, \dots, ih_{16} \in DE_{18(10)}\,.$$

5.4 The Subalgebras DE_{10} and $\mathfrak{so}(16)$

We will now define two more subalgebras of $DE_{18(10)}$. First note that the "black" nodes in the $DE_{18(10)}$ Dynkin diagram (figure 6) define a subalgebra isomorphic to $\mathfrak{so}(16)$ and the "white" nodes define a subalgebra isomorphic to E_{10}. This is depicted in figure 7. Note that E_{10} is only a subalgebra of the complexified DE_{18}, but $\mathfrak{so}(16) \subset DE_{18(10)}$. These two subalgebras intersect trivially, but they do not commute with each other, because the node β_7 of E_{10} is connected to the node β_9 of $\mathfrak{so}(16)$. We denote the commutant of $\mathfrak{so}(16)$ by

$$\mathfrak{g}^{(\text{com})} = \{x \in DE_{18(10)}: \quad [x, y] = 0 \quad \forall y \in \mathfrak{so}(16) \subset DE_{18(10)}\}\,.$$

Proposition 5.1. $\mathfrak{g}^{(\text{com})}$ *contains a* DE_{10} *subalgebra.*

Proof. We begin with the complexified algebra. Recall the root-space decomposition

$$DE_{18} = \bigoplus_{\alpha \in \Delta(DE_{18})} \mathfrak{g}(DE_{18})_\alpha\,.$$

Note that the subspaces $\mathfrak{g}(DE_{18})_{\pm\alpha}$, for $\alpha = \beta_{-1}, \dots, \beta_6, \beta_8$, commute with $\mathfrak{so}(16)$. Now define

$$\chi = (\beta_6 + 2\beta_7) + (2\beta_9 + 2\beta_{10} + 2\beta_{11} + 2\beta_{12} + 2\beta_{13} + 2\beta_{14} + \beta_{15} + \beta_{16})\,. \tag{5.2}$$

Note that the involution ω^* defined in section 5.3 satisfies $\omega^*(\chi) = \chi$. This implies that the coroot $\chi^\vee$ is in the real $DE_{18(10)}$.

Since χ is a root of a finite D_{10} subalgebra (corresponding to the Dynkin sub-diagram that includes only the nodes $\beta_6, \beta_7, \beta_9, \cdots, \beta_{16}$), $\dim \mathfrak{g}(DE_{18})_{\pm\chi} = 1$. It is not hard to see that the elements of $\mathfrak{g}(DE_{18})_{\pm\chi}$ commute with $\mathfrak{so}(16)$ (they can be embedded in $\mathfrak{so}(20) \supset$

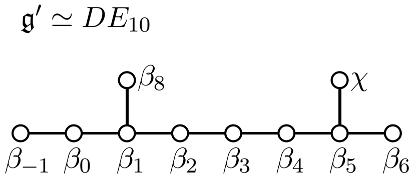

Figure 8: The Dynkin diagram of the subalgebra $\mathfrak{g}' \subset DE_{18}$.

$\mathfrak{so}(16)$). Consider the set of positive real roots $\beta_{-1}, \ldots, \beta_6, \beta_8, \chi$. These roots all square to 2, and their intersection matrix is encoded in the Dynkin diagram of figure 8. This is again the Dynkin diagram of DE_{10}. The rest of the proof is technical and can be found in appendix B.

By now, we have defined three different algebras that contain DE_{10}: the $\mathbb{Z}_2$-invariant subalgebra $\mathfrak{g}^{(\mathrm{inv})} \subset E_{10}$, the $\mathfrak{so}(16)$-commutant subalgebra $\mathfrak{g}^{(\mathrm{com})} \subset DE_{18}$, and the abstract DE_{10} algebra. We need a way to relate them to each other. The embedding of DE_{10} in E_{10} sends the Cartan subalgebra $\mathfrak{h}(DE_{10})$ to the Cartan subalgebra $\mathfrak{h}(E_{10})$. It thus induces a linear isomorphism between the Cartan subalgebra $\mathfrak{h}(DE_{10})$ and $\mathfrak{h}(E_{10})$. Similarly, the embedding of DE_{10} in DE_{18} induces a linear map from $\mathfrak{h}(DE_{10})$ to $\mathfrak{h}(DE_{18})$. Since these maps are going to be important later on, we will now describe them in more detail.

Comparing the Dynkin diagram of figure 2 to that of figure 8, we define a map σ from $\mathfrak{h}(DE_{10})^*$ to $\mathfrak{h}(DE_{18})^*$ that sends the simple roots of DE_{10} to roots of DE_{18} and preserves their inner products. It acts on the generators as follows:

$$\begin{array}{ccccc} \sigma(\gamma_{-1}) = \beta_{-1}\,, & \sigma(\gamma_0) = \beta_0\,, & \sigma(\gamma_1) = \beta_1\,, & \sigma(\gamma_2) = \beta_2\,, & \sigma(\gamma_3) = \beta_3, \\ \sigma(\gamma_4) = \beta_4\,, & \sigma(\gamma_5) = \beta_5\,, & \sigma(\gamma_6) = \beta_6\,, & \sigma(\gamma_7) = \beta_8\,, & \sigma(\gamma_8) = \chi. \end{array} \tag{5.3}$$

σ is the unique injective map that is compatible with the bilinear forms $(\cdot|\cdot)$ on DE_{10} and DE_{18} and satisfies

$$0 = (\sigma(\alpha)|\beta)\,, \qquad \text{for } \alpha \in \mathfrak{h}(DE_{10})^* \text{ and } \beta \in \sum_{j=9}^{16} \mathbb{C}\beta_j \equiv \mathfrak{h}(D_{16})^*.$$

Now we will define a map from $\mathfrak{h}(DE_{18})^*$ to $\mathfrak{h}(DE_{10})^*$. Given $\beta \in \mathfrak{h}(DE_{18})^*$ we first take its projection to the image of σ, and, using σ^{-1}, we identify this projection with an element of $\mathfrak{h}(DE_{10})^*$. Thus, we define a linear map $\varpi' : \mathfrak{h}(DE_{18})^* \to \mathfrak{h}(DE_{10})^*$, that satisfies

$$\varpi' \circ \sigma = I\,, \qquad \varpi'(\beta) = 0 \quad \text{for } \beta \in \sum_{j=9}^{16} \mathbb{C}\beta_j \equiv \mathfrak{h}(D_{16})^*.$$

Explicitly, ϖ' can be defined by

$$\begin{gathered} \varpi'(\beta_{-1}) = \gamma_{-1}\,, \quad \varpi'(\beta_0) = \gamma_0\,, \quad \varpi'(\beta_1) = \gamma_1\,, \quad \varpi'(\beta_2) = \gamma_2\,, \quad \varpi'(\beta_3) = \gamma_3\,, \\ \varpi'(\beta_4) = \gamma_4\,, \quad \varpi'(\beta_5) = \gamma_5\,, \quad \varpi'(\beta_6) = \gamma_6\,, \quad \varpi'(\chi) = \gamma_8\,, \quad \varpi'(\beta_8) = \gamma_7\,, \\ \varpi'(\beta_9) = \varpi'(\beta_{10}) = \cdots = \varpi'(\beta_{16}) = 0\,. \end{gathered} \tag{5.4}$$

Finally, we need a map from $\mathfrak{h}(DE_{18})^*$ back to $\mathfrak{h}(E_{10})^*$. Using (4.8), we define $\varpi : \mathfrak{h}(DE_{18})^* \to \mathfrak{h}(E_{10})^*$ by

$$\varpi := \upsilon \circ \varpi' .$$

Explicitly, ϖ can be defined by

$$\begin{aligned} &\varpi(\beta_{-1}) = \alpha_{-1}\,, \quad \varpi(\beta_0) = \alpha_0\,, \quad \varpi(\beta_1) = \alpha_1\,, \quad \varpi(\beta_2) = \xi\,, \quad \varpi(\beta_3) = \alpha_7\,, \\ &\varpi(\beta_4) = \alpha_6\,, \quad \varpi(\beta_5) = \alpha_5\,, \quad \varpi(\beta_6) = \alpha_4\,, \quad \varpi(\chi) = \alpha_8\,, \quad \varpi(\beta_8) = \alpha_2\,, \\ &\varpi(\beta_9) = \varpi(\beta_{10}) = \cdots = \varpi(\beta_{16}) = 0\,. \end{aligned} \tag{5.5}$$

Using (5.5) and (5.2), we get

$$\varpi(\beta_7) = \frac{1}{2}(\varpi(\chi) - \varpi(\beta_6)) = \frac{1}{2}(\alpha_8 - \alpha_4)\,. \tag{5.6}$$

Given a root $\beta \in \Delta(DE_{18}) \subset \mathfrak{h}(DE_{18})^*$ we define its *physical action* by,

$$I(\beta) := 2\pi e^{\langle \varpi(\beta), \vec{h} \rangle}\,, \tag{5.7}$$

where $\vec{h} \in \mathfrak{h}(E_{10})$ is a linear function of the log's of the compactification radii, as defined in (2.6). We will also denote by

$$\varrho : \mathfrak{h}(D_8)^* \to \mathfrak{h}(DE_{18})^* \tag{5.8}$$

the linear map that sends the simple roots $\beta_9, \dots, \beta_{16}$ of D_8 to the simple roots $\beta_9, \dots, \beta_{16}$ of DE_{18}.

The linear maps that we have defined so far are collected in the following line

$$\mathfrak{h}(D_8)^* \xrightarrow{\varrho} \mathfrak{h}(DE_{18})^* \xrightarrow{\varpi'} \mathfrak{h}(DE_{10})^* \xrightarrow{\upsilon} \mathfrak{h}(E_{10})^* \xrightarrow{\upsilon^{-1}} \mathfrak{h}(DE_{10})^* \xrightarrow{\sigma} \mathfrak{h}(DE_{18})^* \qquad \varpi = \upsilon \circ \varpi' .$$

(Note that this is not an exact sequence!)

We note the analogous statements for M-theory on T^8. If we replace M-theory on T^{10} with M-theory on T^8, we have to replace $DE_{18(10)}$ with $\mathfrak{so}(24,8)$, and DE_{10} with $\mathfrak{so}(8,8)$. The $E_{10} \subset DE_{18}$ would be analogous to $A_8 = sl(9,\mathbb{R})$, which is the subalgebra corresponding to the "white" nodes of the Dynkin diagram of $D_{16(8)}$.

It is interesting to compare $\mathfrak{g}^{(\mathrm{inv})}$ to $\mathfrak{g}^{(\mathrm{com})}$, since both contain a DE_{10} subalgebra, and both have $\mathfrak{h}(DE_{10})$ as a maximal abelian subalgebra. Both $\mathfrak{g}^{(\mathrm{inv})}$ and $\mathfrak{g}^{(\mathrm{com})}$ have a root-space decomposition with respect to $\mathfrak{h}(DE_{10})$ and the root-spaces of both algebras are identical to $\Delta(DE_{10})$.

It turns out that $\mathfrak{g}^{(\mathrm{com})}$ is strictly bigger than DE_{10}, and this can be seen already at level-1, namely, some multiplicities of level-1 roots of $\mathfrak{g}^{(\mathrm{inv})}$ are smaller than the corresponding $\mathfrak{g}^{(\mathrm{com})}$ multiplicities. The question of whether $\mathfrak{g}^{(\mathrm{com})}$ contains $\mathfrak{g}^{(\mathrm{inv})}$ or not will be studied in another paper.

JHEP06(2005)057

5.5 The maximal compact subalgebra $\mathfrak{k}(DE_{18(10)})$

We need to define a "maximal compact subalgebra" $\mathfrak{k}(DE_{18(10)}) \subset DE_{18(10)}$. (If we replace T^{10} with T^{8}, for comparison, $\mathfrak{k}$ will become $\mathfrak{so}(24)\times\mathfrak{so}(8)$.) To define the maximal compact subalgebra, we first define the *compact involution* ω_c on $DE_{18(10)}$ as follows. We define it on the Chevalley generators (see section 5.3) by

$$\omega_c(h_i) = -h_i\,, \quad \omega_c(e_i) = f_i\,, \quad \omega_c(f_i) = e_i\,, \qquad \text{for } i = -1\ldots 16.$$

We then define

$$\mathfrak{k}(DE_{18(10)}) = \{x \in DE_{18(10)}:\ \ \omega_c(x) = x\}\,.$$

Note that

$$ih_j, e_j + f_j, i(e_j - f_j) \in \mathfrak{k}(DE_{18(10)}) \qquad \text{for } j = 9,\ldots,16.$$

These generators generate a real orthogonal subalgebra $\mathfrak{so}(16) \subset \mathfrak{k}(DE_{18(10)})$. In general, little is known about such "maximal compact" subalgebras of infinite dimensional Kac-Moody algebras (but see [54] for a recent study of $\mathfrak{k}(E_9)$). The physical degrees of freedom of M-theory on $T^{10}/\mathbb{Z}_2$ are presumed to be related to the coset of Lie algebras $DE_{18(10)}/\mathfrak{k}(DE_{18(10)})$.

6. The twisted sector

We are now ready to discuss the twisted sector of the orbifold. As in section 4, we restrict attention to the orbifold $T^5 \times (T^5/\mathbb{Z}_2)$ from section 3.1. The twisted sector consists of 16 M5-branes which support fluxes and particle charges. Boundaries of M2-branes attached to the M5-branes are the origin for these charges. Experience with E_{10} suggests that fluxes are related to real roots and particle charges are related to imaginary (isotropic) roots of the Kac-Moody algebra. We would like to extend this type of relation to the twisted sectors.

We have seen in section 5.4 that DE_{18} has a $D_8 \simeq \mathfrak{so}(16)$ subalgebra. This number 16 that appears in $\mathfrak{so}(16)$ is directly related to the 16 M5-branes as follows. The Lie algebra DE_{18} can be decomposed into (an infinite number of) irreducible representations of $\mathfrak{so}(16) \subset DE_{18}$. We will propose below that the twisted sector fluxes and charges can be matched with such irreducible $\mathfrak{so}(16)$ representations.

6.1 $\mathfrak{so}(16)$-packs

In section 5.4 we identified a $DE_{10} \subset DE_{18}$ subalgebra. The Cartan subalgebra $\mathfrak{h}(DE_{10})$ is identified with

$$\mathbb{C}\beta_{-1}^\vee + \mathbb{C}\beta_0^\vee + \mathbb{C}\beta_1^\vee + \mathbb{C}\beta_2^\vee + \mathbb{C}\beta_3^\vee + \mathbb{C}\beta_4^\vee + \mathbb{C}\beta_5^\vee + \mathbb{C}\beta_6^\vee + \mathbb{C}\beta_8^\vee + \mathbb{C}\chi^\vee\,,$$

(where $\beta_i^\vee$ ($i = -1\ldots 16$) are the simple coroots, and $\chi^\vee$ is the coroot dual to χ). It commutes with $\mathfrak{so}(16)$ and is isomorphic to $\mathbb{C}^{10}$. We can therefore decompose DE_{18} under its $\mathfrak{h}(DE_{10}) \oplus \mathfrak{so}(16)$ subalgebra. The irreducible representations of $\mathfrak{h}(DE_{10}) \oplus \mathfrak{so}(16)$ that appear in this decomposition will be called $\mathfrak{so}(16)$-*packs*.

It is slightly more convenient to work with $\mathfrak{h}(E_{10})$ rather than $\mathfrak{h}(DE_{10})$. We switch to $\mathfrak{h}(E_{10})$ using the map $\upsilon : \mathfrak{h}(DE_{10})^* \to \mathfrak{h}(E_{10})^*$ defined in (4.8). We now have the decomposition

$$DE_{18} = \bigoplus_{\tilde{\lambda}} \tilde{\mathfrak{g}}_{\tilde{\lambda}}\,, \qquad \text{where } \tilde{\lambda} \text{ labels irreps of } \mathfrak{so}(16) \oplus \varpi^*(\mathfrak{h}(E_{10})). \tag{6.1}$$

Here ϖ is the map from (5.5) that allows us to translate an E_{10} root to a DE_{18} root.

An irreducible representation of $\varpi^*(\mathfrak{h}(E_{10})) \simeq \mathfrak{h}(E_{10})$ is 1-dimensional and is labeled by an element $\lambda \in \mathfrak{h}(E_{10})^*$. Moreover, if $\beta \in \Delta(DE_{18})$ then $\varpi'(\beta) \in P(DE_{10})$, where $P(DE_{10})$ was defined in (4.7), and ϖ' is the map from (5.4). The irreducible representations of $\mathfrak{so}(16) \oplus \varpi^*(\mathfrak{h}(E_{10}))$ that appear in (6.1) are therefore labeled by a pair

$$\tilde{\lambda} = (\lambda, \mathbf{r})\,, \qquad \text{where } \lambda \in \upsilon(P(DE_{10})), \text{ and } \mathbf{r} \text{ is an irrep of } \mathfrak{so}(16).$$

We call such a pair an $\mathfrak{so}(16)$-*pack*. The weight λ will eventually describe the dependence of the physical object associated with $\tilde{\lambda}$ on the radii $R_1, \ldots, R_{10}$, and $\mathbf{r}$ will determine the object's relation to the 16 M5-branes.

The representations $\mathbf{r}$ are all finite. They are completely determined by their lowest (or highest) D_8 weight $\tilde{\tilde{\Lambda}} \in P(D_8)$. Let $x \in \tilde{\mathfrak{g}}_{\tilde{\lambda}} \subset DE_{18}$ be a vector with D_8 lowest-weight $\tilde{\tilde{\Lambda}}$. Then x belongs to a single root-space $(DE_{18})_\beta$, and the root β of DE_{18} can be calculated as follows:

$$\beta = \sigma(\lambda) + \varrho(\tilde{\tilde{\Lambda}})\,.$$

(See (5.3) and (5.8) for the definitions of the linear maps σ and ϱ.) Alternatively, if the DE_{18}-root β is given, then the $\mathfrak{so}(16)$-representation of which x is a lowest-weight vector can be found by calculating the lowest weight $\tilde{\tilde{\Lambda}}$:

$$\varrho(\tilde{\tilde{\Lambda}}) = \beta - \sigma(\varpi'(\beta))\,. \tag{6.2}$$

For $\tilde{\tilde{\Lambda}}$ to be a lowest-weight vector, it must satisfy

$$0 \geq (\varrho(\tilde{\tilde{\Lambda}})|\beta_j)\,, \qquad j = 9, \ldots, 16\,.$$

We denote the fundamental weights of D_8 by $\tilde{\tilde{\Lambda}}_9, \ldots, \tilde{\tilde{\Lambda}}_{16} \in P(D_8)$. This notation is compatible with the simple roots of D_8 that we take to be $\beta_9, \ldots, \beta_{16}$.

Now take γ to be a DE_{18}-root with support only on the white nodes of figure 6,

$$\gamma = \sum_{i=-1}^{8} k_i \beta_i\,.$$

Take an element $x \in \mathfrak{g}(DE_{18})_\gamma$ and complete it to an $\mathfrak{so}(16)$ representation by acting on it with all possible combinations of the Chevalley generators $e_9, f_9, h_9, \ldots, e_{16}, f_{16}, h_{16}$ (that generate $\mathfrak{so}(16) \subset DE_{18}$). This is denoted by $U(\mathfrak{so}(16))x$, where $U(\mathfrak{so}(16))$ is the universal enveloping algebra of $\mathfrak{so}(16)$. It is easy to check that $(\gamma|\beta_j) = (-k_7\tilde{\tilde{\Lambda}}_9|\beta_j)$ for all the simple roots β_j ($j = 9, \ldots, 16$) of $\mathfrak{so}(16)$. Without loss of generality, suppose $k_7 > 0$. Then $[f_j, \gamma] = 0$ for $j = 9, \ldots, 16$ and therefore $U(\mathfrak{so}(16))x$ is a lowest-weight representation with lowest-weight $-k_7\tilde{\tilde{\Lambda}}_9$.

Proposition 6.1. *Let* $\gamma = \sum_{i=-1}^{8} k_i\beta_i$ *be a root of* DE_{18} *and let* $0 \neq x \in \mathfrak{g}_\gamma$. *Then* $U(\mathfrak{so}(16))x \simeq L_{\mathfrak{so}(16)}(|k_7|\tilde{\tilde{\Lambda}}_9)$ *where* $L_{\mathfrak{so}(16)}(\tilde{\tilde{\Lambda}})$ *is the irreducible* $\mathfrak{so}(16)$*-representation with highest-weight* $\tilde{\tilde{\Lambda}}$. *In particular,*

a. *If* $k_7 = 0$, *then* $U(\mathfrak{so}(16))x$ *is equivalent to the* 1*-dimensional singlet representation* **1**.

b. *If* $k_7 = \pm 1$, *then* $U(\mathfrak{so}(16))x$ *is equivalent to the* 16*-dimensional vector representation* **16**.

c. *If* $k_7 = \pm 2$, *then* $U(\mathfrak{so}(16))x$ *is equivalent to the traceless symmetric tensors* **135**.

The proof can be found in appendix C.

Of course, not all $\mathfrak{so}(16)$-packs are of the form described in proposition 6.1. The support of the root χ (from (5.2)), for example, is not restricted to the "white" nodes, and it is a highest (lowest) weight of a singlet $\mathfrak{so}(16)$-representation **1**.

6.2 Examples of $\mathfrak{so}(16)$-packs

We will now give a few examples of $\mathfrak{so}(16)$-packs. Let $\tilde{\lambda} = (\lambda, \mathbf{r})$ be an $\mathfrak{so}(16)$-pack. It is more convenient to present $\upsilon(\lambda)$ instead of λ, and it is convenient to write $\upsilon(\lambda) \in \mathfrak{h}(E_{10})^*$ in the basis (2.4) [υ was defined in (4.8)]. For small representations $\mathbf{r}$, it is convenient to present the dimension $\dim \mathbf{r}$. Thus, we will present an $\mathfrak{so}(16)$-pack as

$$\tilde{\lambda} = (\tilde{n}_1, \ldots, \tilde{n}_{10}; \dim \mathbf{r}), \qquad \tilde{n}_j \in \tfrac{1}{2}\mathbb{Z}, \quad (j = 1, \ldots, 10). \tag{6.3}$$

Note that from (2.3) and (5.5)-(5.6) it follows that $\tilde{n}_j$ $(j = 1 \ldots 10)$ could be integers or half-integers.

Using proposition 6.1, we can make a list of some real DE_{18} roots that are $\mathfrak{so}(16)$-lowest-weights. Using (5.5)–(5.6) and (2.4), we can translate the roots to the physical notation (6.3). We get,

$$\begin{aligned}
\beta_7 &\to \left(0,0,0,0,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};\mathbf{16}\right),\\
\sum_{j=6}^{7}\beta_j &\to \left(0,0,0,0,0,\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};\mathbf{16}\right),\\
\sum_{j=5}^{7}\beta_j &\to \left(0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2};\mathbf{16}\right),\\
\sum_{j=4}^{7}\beta_j &\to \left(0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2};\mathbf{16}\right),\\
\sum_{j=3}^{7}\beta_j &\to \left(0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2};\mathbf{16}\right).
\end{aligned} \tag{6.4}$$

In general, it is obvious that given an $\mathfrak{so}(16)$-pack in the form

$$(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}; \mathbf{r}),$$

we can generate more $\mathfrak{so}(16)$-packs by reshuffling $n_1, \dots, n_5$ and reshuffling $n_6, \dots, n_{10}$. That is, for every pair of permutations $(\sigma, \sigma') \in S_5 \times S_5$,

$$(n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}, n_{\sigma(4)}, n_{\sigma(5)}, n_{5+\sigma'(1)}, n_{5+\sigma'(2)}, n_{5+\sigma'(3)}, n_{5+\sigma'(4)}, n_{5+\sigma'(5)}; \mathbf{r}),$$

is also an $\mathfrak{so}(16)$-pack. Physically, $S_5 \times S_5$, which acts by permuting the radii, is clearly a symmetry of the problem. Mathematically, $S_5 \times S_5$ is a subgroup of the Weyl group of DE_{18}. It is therefore sufficient — as we will do from now on — to list only one $\mathfrak{so}(16)$-pack from each $S_5 \times S_5$ multiplet.

The next $\binom{5}{2}$ $\mathfrak{so}(16)$-packs in our examples are $S_5 \times S_5$ permutations of

$$\sum_{j=2}^{7} \beta_j \to \left(0, 0, 0, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \mathbf{16}\right). \tag{6.5}$$

Finally, we will need the 5×5 permutations of

$$\beta_0 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 \to \left(0, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \mathbf{16}\right). \tag{6.6}$$

7. Physical interpretation

We will now propose a physical interpretation for $\mathfrak{so}(16)$-packs. The vector $\lambda = (\tilde{n}_1, \dots, \tilde{n}_{10})$ that appears in (6.3) is an element of $\mathfrak{h}(E_{10})^*$ and it is therefore natural to examine the inner product $\langle \lambda, \vec{h} \rangle$, where $\vec{h}$ is the vector of logs of radii, defined in (2.6).

$$I(\lambda) {:=} 2\pi e^{\langle \lambda, \vec{h} \rangle} = 2\pi \prod_{i=1}^{10} (M_p R_i)^{\tilde{n}_i},$$

can be interpreted as an "action." This definition is consistent with (5.7) with $\lambda = \varpi(\beta)$ for any root β of the $\mathfrak{so}(16)$-pack.

The $\mathfrak{so}(16)$-representation $\mathbf{r}$ that appears in (6.3) can tell us the origin of the physical object in relation to the 16 M5-branes. For example, $\mathbf{r} = \mathbf{16}$ suggests that the object is related to a single M5-brane. Below we will study the physical interpretation of $\mathfrak{so}(16)$-packs in several examples. We will begin with $\mathfrak{so}(16)$-packs that can be interpreted as fluxes, and we will end with more complicated examples that can be interpreted as "charges."

7.1 Fluxes of the twisted sector

The twisted-sector provides more varieties of fluxes. Consider, for concreteness, the orbifold $T^5 \times (T^5/\mathbb{Z}_2)$ from section 3.1. Let the radii of the T^5 factor be $R_1, \dots, R_5$ and the radii of the $(T^5/\mathbb{Z}_2)$ factor be $R_6, \dots, R_{10}$. The twisted sector consists of 16 M5-branes [10], and each M5-brane carries a 5+1D tensor multiplet of $N = (2, 0)$ supersymmetry. The bosonic fields of the tensor multiplet are an anti-self-dual tensor field and 5 scalar fields. The 5 scalar fields can be interpreted as the coordinates of the M5-brane in the $T^5/\mathbb{Z}_2$ directions.

We will assume that we are in the asymptotic region of "moduli-space," (2.8). In this region, the M5-branes are very heavy, and a nonrelativistic limit is a good approximation in many cases. For example, with k units of momentum in, say, the 6^{th} direction, an

M5-brane has a kinetic energy which, in the nonrelativistic limit (2.8), is

$$\frac{1}{2M_p^6 R_1 \cdots R_5}\left(\frac{k}{R_6}\right)^2 . \tag{7.1}$$

We need to measure energy with respect to conformal time, which means an extra factor of $2\pi M_p^9 R_1 \cdots R_{10}$. Multiplying (7.1) by this factor, we get

$$\Phi = \pi M_p^3 k^2 \frac{R_7 R_8 R_9 R_{10}}{R_6} . \tag{7.2}$$

Similarly, we get four more fluxes corresponding to M5-brane momenta in the $7^{th}, \dots, 10^{th}$ directions.

The anti-self-dual 3-form field strength H_{IJK} $(I = 0, \dots, 5)$ of the tensor multiplet will give us 10 more possible fluxes, with energies proportional to

$$\pi M_p^9 R_1^2 R_2^2 R_6 R_7 R_8 R_9 R_{10}, \dots, \pi M_p^9 R_4^2 R_5^2 R_6 R_7 R_8 R_9 R_{10}, \tag{7.3}$$

in units of conformal time. We will now relate these fluxes to $\mathfrak{so}(16)$-packs.

7.2 $\mathfrak{so}(16)$-packs for Fluxes

Recall that real roots of E_{10} are related to fluxes, and the squared action $\exp(2\langle\alpha, \vec{h}\rangle)$ of a real root α is the energy stored in the flux, measured in units conjugate to conformal time [20, 55, 56, 1]. In other words,

$$E \propto N^2 \frac{e^{2\langle\alpha,\vec{h}\rangle}}{M_p^9 \prod_{j=1}^{10} R_j}$$

is the energy stored in the flux, where $N \in \mathbb{Z}$ is the number of flux units. We will now extend this formula to the twisted-sectors.

The $\mathfrak{so}(16)$-packs from (6.4)-(6.5) should be associated with the fluxes (7.2)–(7.3). Using (5.6), the flux energy (7.2) can be written as $\pi k^2 e^{2\langle\varpi(\beta_7),\vec{h}\rangle}$. We therefore associate this flux to β_7, which is the lowest-weight of the first $\mathfrak{so}(16)$-pack in (6.4). Similarly, all the other $\mathfrak{so}(16)$-packs from (6.4) correspond to fluxes of the scalar components of the 16 tensor multiplets.

The roots from (6.5) match the fluxes of the anti-self-dual tensor field (7.3). For example,

$$N^2 M_p^9 R_4^2 R_5^2 R_6 R_7 R_8 R_9 R_{10} = N^2 e^{2\langle\varpi(\sum_{j=2}^{7}\beta_j),\vec{h}\rangle} .$$

Now consider the 5×5 $\mathfrak{so}(16)$-packs which are $S_5 \times S_5$ permutations of (6.6). The $\mathfrak{so}(16)$-pack from (6.6) has energy

$$E = \frac{1}{2} N^2 \frac{M_p^6 R_2 R_3 R_4 R_5 R_{10}^2}{R_1} . \tag{7.4}$$

This flux has the following interpretation.

Let X^I $(I = 5, \dots, 10)$ be the 5 scalars of the tensor-multiplet on one of the 16 exceptional M5-branes. The indices $I = 5, \dots, 10$ are chosen to match the directions of the $T^5/\mathbb{Z}_2$. Now consider a constant flux $\partial_1 X^{10}$. Geometrically, this can be described as an exceptional M5-brane that wraps the diagonal of the torus in the 1^{st} and 10^{th} directions. Note that the M5-brane is a point in the $6^{th},\ldots,9^{th}$ directions. The moduli space for X^I is actually $T^5/\mathbb{Z}_2$, but for fixed $X^6,\ldots,X^9$, away from the fixed points of the $\mathbb{Z}_2$-action, $T^5/\mathbb{Z}_2$ looks like an S^1 fibration (corresponding to, say, the 10^{th} direction) over $T^4/\mathbb{Z}_2$ (corresponding to the $6^{th},\ldots,9^{th}$ directions). The fibration does not preserve the orientation of the fiber, and therefore winding number (of the exceptional M5-brane) in the 10^{th} direction is not conserved. This is consistent with the fact that we projected out the imaginary roots

$$(1,2,2,2,2,1,1,1,1,2;\mathbf{1})\,,$$

which would correspond to an M5-brane wrapping the $2^{nd},3^{rd},4^{th},5^{th}$, and 10^{th} directions. However, the constant flux $\partial_1 X^{10}$ is still allowed. The nonconservation of the winding number implies that transitions between zero and nonzero $\partial_1 X^{10}$ flux are possible.

The energy of a wrapped brane is

$$E = M_p^6 R_2 R_3 R_4 R_5 \sqrt{R_1^2 + N^2 R_{10}^2}\,.$$

In the asymptotic region (2.8), the excess energy stored in the $\partial_1 X^{10}$ flux can be written as (7.4), approximately.

The $\mathfrak{so}(16)$-packs (6.4)–(6.6), which we discussed so far, correspond to real roots of DE_{18}. Take β_7, for example. It is a lowest weight of an $\mathfrak{so}(16)$-pack ($\mathfrak{so}(16)$ irreducible representation) that can be decomposed into DE_{18} weight spaces. The $\mathfrak{so}(16)$-pack is isomorphic to the fundamental representation **16**, and, as is easy to check, all its DE_{18}-weights can be generated by acting on β_7 with simple reflections around $\beta_9,\ldots,\beta_{16}$. Thus, all the DE_{18}-weights corresponding to this $\mathfrak{so}(16)$-pack square to 2 and are therefore real roots of DE_{18}. The same statement is true for all the other $\mathfrak{so}(16)$-packs from (6.4)-(6.6). Next, we will discuss $\mathfrak{so}(16)$-packs that contain imaginary roots.

7.3 Imaginary roots

An isotropic root α is an imaginary root that squares to zero. We are interested in prime isotropic roots, i.e. roots that are not a nontrivial integer multiple of another root. Such roots were found in [1] to correspond to brane charges.

According to [25, proposition 5.7], for any simply-laced Kac-Moody algebra, a root α is isotropic if and only if it is W-equivalent to an imaginary root β such that $\operatorname{supp}\beta$ (the set of nodes of the Dynkin diagram whose corresponding simple roots have a nonzero coefficient in β) is a subdiagram of affine type.

We will now apply this to DE_{18}. There are two distinct subdiagrams of affine type. One subdiagram is E_9 and the other is $\widehat{D}_{16}$. $\widehat{D}_{16}$ is obtained by dropping the node β_{-1} (see figure 5), and E_9 is obtained by dropping nodes $\beta_7,\beta_9,\ldots,\beta_{16}$. The positive prime isotropic root with support in E_9 is given by

$$\delta_1 = 2\beta_{-1} + 4\beta_0 + 6\beta_1 + 5\beta_2 + 4\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6 + 3\beta_8\,. \tag{7.5}$$

Its multiplicity is $\operatorname{mult}\delta_1 = 8$, as in E_9. The positive prime isotropic root with support in $\widehat{D}_{16}$ is given by

$$\delta_2 = \beta_0 + \beta_8 + \beta_{15} + \beta_{16} + 2\sum_{i=1}^{7}\beta_i + 2\sum_{i=9}^{14}\beta_i \,. \tag{7.6}$$

Its multiplicity is $\operatorname{mult}\delta_2 = 16$, as in $\widehat{D}_{16}$. It follows that δ_1 and δ_2 are not $W(DE_{18})$-equivalent. Note that

$$\varpi(\delta_1) = 2\alpha_{-1} + 4\alpha_0 + 6\alpha_1 + 8\alpha_2 + 10\alpha_3 + 11\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7 + 5\alpha_8 = \tau_{M5}\,, \tag{7.7}$$

where we used (3.2). Thus, δ_1 corresponds to the charge of an M5-brane from the twisted sector. But note that δ_1 is a $\mathbb{Z}_2$-even root and belongs to the untwisted sector. (There is of course no contradiction, since δ_1 would correspond to an extra M5-brane parallel to the already existing 16 M5-branes.) Note also that

$$\varpi(\delta_2) = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = \delta\,, \tag{7.8}$$

where δ was defined in (4.14) and is identified with the charge of a Kaluza-Klein particle in the 1^{st} direction. It is also a member of the untwisted sector.

In [1], a strategy to understand the meaning of imaginary roots γ was to express them as a sum of real roots, $\gamma = \alpha+\beta$, and check what happens when both fluxes associated with α and β are present. We will now employ a similar strategy for imaginary $\mathfrak{so}(16)$-packs.

The analogs of α, β are two real $\mathfrak{so}(16)$-packs

$$\tilde{\lambda} = (\lambda, \mathbf{r}), \qquad \tilde{\lambda}' = (\lambda', \mathbf{r}')\,.$$

The analog of γ is an $\mathfrak{so}(16)$-pack

$$\tilde{\lambda}'' = (\lambda + \lambda'; \mathbf{r}'')\,, \tag{7.9}$$

such that the $\mathfrak{so}(16)$-representation $\mathbf{r}''$ appears in the decomposition of the tensor product $\mathbf{r}\otimes\mathbf{r}'$. If $\tilde{\lambda}$ and $\tilde{\lambda}'$ can both be associated with fluxes, $\tilde{\lambda}''$ should be associated with an anomalous charge that is present when the two fluxes overlap. We will now demonstrate these ideas in explicit examples.

Wess-Zumino interactions. Take

$$\tilde{\lambda}'' = \left(2, 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; \mathbf{16}\right),$$

(with lowest-weight $\gamma = 2\beta_{-1} + 3\beta_0 + 4\beta_1 + 3\beta_2 + 3\beta_3 + 3\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7 + 2\beta_8 + \chi$), and split it, as in (7.9), into

$$\tilde{\lambda} = (2, 1, 1, 0, 0, 1, 1, 1, 1, 1; \mathbf{1})\,,$$

(with lowest-weight $2\beta_{-1} + 3\beta_0 + 4\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + \beta_6 + \chi + 2\beta_8$),

$$\tilde{\lambda}' = \left(0, 0, 0, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \mathbf{16}\right),$$

(with lowest-weight $\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7$). To analyze this configuration of fluxes more conveniently, we formally reduce from M-theory to type-IIA. For fixed $6^{th}, 7^{th}, 8^{th}, 9^{th}, 10^{th}$ coordinates, we reduce to type-IIA with the 1^{st} circle taken as the M-theory "11^{th}" direction; the exceptional M5-branes become D4-branes; the $\tilde{\lambda}$-flux becomes an RR 2-form field-strength $\mathcal{F}^{\mathrm{RR}}_{45}$ (keeping the same notation for directions as for M-theory), and $\tilde{\lambda}'$ corresponds to a field-strength F_{23} of the gauge field on the D4-brane. We need to assume that $M_p R_1 \ll 1$ for this reduction, and therefore relax the condition (2.7). Since we are dealing with discrete charges, however, the final conclusion will be valid in the limit (2.7).

Suppose there are N units of $\mathcal{F}^{\mathrm{RR}}_{45}$-flux and N' units of F_{23}-flux. The Wess-Zumino interaction $A \wedge F \wedge \mathcal{F}^{\mathrm{RR}}$ on the D4-brane world-volume [57, 58] now produces, in the presence of the $\tilde{\lambda}$ and $\tilde{\lambda}'$ fluxes, a background charge for the D4-brane gauge field A. This charge must be cancelled by NN' open strings that end on the D4-brane. Elevating back to M-theory, we see that these strings become open M2-branes with one end on one of the exceptional M5-branes; the other end could be at one of the 32 $\mathbb{Z}_2$ fixed-hyperplanes (which extend in the $1^{st}, \ldots, 5^{th}$ directions and run parallel to the 16 M5-branes). We conclude that $\tilde{\lambda}''$ is associated with the charge of open M2-branes that end on the exceptional M5-branes. Note that the lowest-weight γ is an imaginary root of DE_{18} and it satisfies $\gamma^2 = 0$. In fact,

$$\gamma = 2\beta_{-1}+3\beta_0+4\beta_1+3\beta_2+3\beta_3+3\beta_4+3\beta_5+3\beta_6+3\beta_7+2\beta_8+\beta_{15}+\beta_{16}+2\sum_{j=1}^{6}\beta_{8+j} \quad (7.10)$$

It is not hard to check that γ is $W(DE_{18})$-equivalent to δ_2 from (7.6). Its multiplicity, as a DE_{18} root, is therefore

$$\operatorname{mult}\gamma = \operatorname{mult}\delta_2 = 16.$$

We do not know yet what the physical significance of the multiplicity is.

Scalar-field fluxes. Now take the following $\mathfrak{so}(16)$-pack

$$\tilde{\lambda}'' = \left(2, 2, 2, 2, 2, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; \mathbf{16}\right)$$

(with lowest-weight $2\beta_{-1} + 4\beta_0 + 6\beta_1 + 5\beta_2 + 4\beta_3 + 3\beta_4 + 2\beta_5 + \beta_6 + \beta_7 + 3\beta_8$), and split it into

$$\tilde{\lambda} = (2, 1, 1, 1, 1, 0, 1, 1, 1, 0; \mathbf{1})$$

(with lowest-weight $2\beta_{-1} + 3\beta_0 + 4\beta_1 + 3\beta_2 + 3\beta_3 + 2\beta_4 + \beta_5 + 2\beta_8$),

$$\tilde{\lambda}' = \left(0, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \mathbf{16}\right)$$

(with lowest-weight $\beta_0 + 2\beta_1 + 2\beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8$). The flux associated with $\tilde{\lambda}$ is a nonzero first Chern class c_1 for the fibration of the circle in the 1^{st} direction over the T^2 that is made of the circles in the 6^{th} and 10^{th} directions. The relevant part of the metric is

$$R_1^2\left(dx_1 - \frac{N}{2\pi}x_6 dx_{10}\right)^2 + R_6^2 dx_6^2 + R_{10}^2 dx_{10}^2 \,.$$

The flux associated with $\tilde{\lambda}'$ was discussed after (7.4) and corresponds to $N' = \partial_1 X^{10}$. Here, X^{10} is the 10^{th} coordinate of one of the exceptional M5-branes.

JHEP06(2005)057

DE_{10}

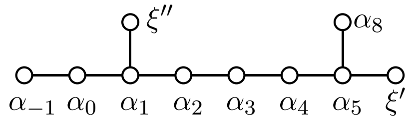

Figure 9: The Dynkin diagram of the $\mathbb{Z}_2$-invariant subalgebra $DE_{10} \subset E_{10}$ that is associated with the type-IA orbifold.

The effect we are looking for comes from a cubic coupling of the form

$$-NM_p^6 R_1 R_2 R_3 R_4 R_5 X^6 \partial_1 X^{10} = -NN' M_p^6 R_1 R_2 R_3 R_4 R_5 X^6 \,, \tag{7.11}$$

on the M5-brane world-volume. (The term "cubic" refers to the powers of N and X's together.) This term can be inferred by expanding the Nambu-Goto action for an M5-brane, which in our case reads:

$$\begin{aligned} & M_p^6 R_2 R_3 R_4 R_5 \sqrt{G_{11} + 2G_{1,10}\partial_1 X^{10} + G_{10,10}(\partial_1 X^{10})^2} \approx \\ & \approx M_p^6 R_2 R_3 R_4 R_5 (\sqrt{G_{11}} + \frac{G_{1,10}}{\sqrt{G_{11}}}\partial_1 X^{10}) \\ & \approx M_p^6 R_2 R_3 R_4 R_5 \left(\sqrt{G_{11}} + \left.\frac{G_{1,10}}{\sqrt{G_{11}}}\right|_0 \partial_1 X^{10} + \partial_6 \left.\left(\frac{G_{1,10}}{\sqrt{G_{11}}}\right)\right|_0 X^6 \partial_1 X^{10} + \cdots \right), \end{aligned} \tag{7.12}$$

where G_{IJ} $(I, J = 1 \ldots, 10)$ is the target-space metric. It is the last term in (7.12) that is responsible for the cubic term (7.11). The term (7.11) is linear in the field X^6, and hence is a source for that field. We conclude that the $\mathfrak{so}(16)$-pack $\tilde{\lambda}''$ is associated with a source term for X^6. It would be interesting to look for the implications of other nonlinear terms in the effective action of the M5-brane [59] (and see also [60]).

8. Type IA

For completeness, we will now present some examples of $\mathfrak{so}(16)$-packs for type-IA string theory. The corresponding E_{10} orbifold was discussed in section 3.4. The $\mathbb{Z}_2$-charge is $k_6 + k_7$. A basis for the sublattice of $Q(E_{10})$ generated by $\mathbb{Z}_2$-invariant roots is given by

$$\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_8, \xi', \xi''$$

where

$$\begin{aligned} \xi' &= \alpha_6 + \alpha_7 \to (0,0,0,0,0,0,0,1,0,-1), \\ \xi'' &= \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 3\alpha_6 + \alpha_7 + 2\alpha_8 \to (0,0,0,1,1,1,1,1,0,1). \end{aligned}$$

The inner products are given by Instead of (4.8), we define a linear map $\upsilon_{IA} : \mathfrak{h}(DE_{10})^* \to \mathfrak{h}(E_{10})^*$ by

$$\upsilon_{IA}(\gamma_{-1}) = \alpha_{-1}\,, \quad \upsilon_{IA}(\gamma_0) = \alpha_0\,, \quad \upsilon_{IA}(\gamma_1) = \alpha_1\,, \quad \upsilon_{IA}(\gamma_2) = \alpha_2\,, \quad \upsilon_{IA}(\gamma_3) = \alpha_3\,,$$

$$\upsilon_{IA}(\gamma_4) = \alpha_4\,, \quad \upsilon_{IA}(\gamma_5) = \alpha_5\,, \quad \upsilon_{IA}(\gamma_6) = \xi'\,, \quad \upsilon_{IA}(\gamma_7) = \xi''\,, \quad \upsilon_{IA}(\gamma_8) = \alpha_8\,. \tag{8.1}$$

Instead of (5.5), we define

$$\varpi_{IA} := \upsilon_{IA} \circ \varpi'\,,$$

which gives, using (8.1) and (5.4),

$$\begin{aligned}
&\varpi_{IA}(\beta_{-1}) = \alpha_{-1}\,, \quad \varpi_{IA}(\beta_0) = \alpha_0\,, \quad \varpi_{IA}(\beta_1) = \alpha_1\,, \quad \varpi_{IA}(\beta_2) = \alpha_2\,, \quad \varpi_{IA}(\beta_3) = \alpha_3,\\
&\varpi_{IA}(\beta_4) = \alpha_4\,, \quad \varpi_{IA}(\beta_5) = \alpha_5\,, \quad \varpi_{IA}(\beta_6) = \xi'\,, \quad \varpi_{IA}(\chi) = \alpha_8\,, \quad \varpi_{IA}(\beta_8) = \xi'',\\
&\varpi_{IA}(\beta_9) = \varpi_{IA}(\beta_{10}) = \cdots = \varpi_{IA}(\beta_{16}) = 0\,.
\end{aligned}$$

Using (5.2), we get

$$\varpi_{IA}(\beta_7) = \frac{1}{2}(\varpi_{IA}(\chi) - \varpi_{IA}(\beta_6)) = \frac{1}{2}(\alpha_8 - \alpha_6 - \alpha_7)\,.$$

In type-IA the twisted sectors are D8-branes. We will now analyze the Wess-Zumino interactions on D8-branes, in a fashion similar to section 7.3.

Take for example the Wess-Zumino interaction $A_0 \wedge F_{12} \wedge \mathcal{F}^{\mathrm{RR}}_{345678}$, where again $\mathcal{F}^{\mathrm{RR}}$ is a Ramond-Ramond 6-form field-strength, A is the 1-form Yang-Mills gauge field on a D8-brane, and $F = dA$ is its field-strength. The flux F_{12} is associated to the $\mathfrak{so}(16)$-pack

$$\begin{aligned}
\tilde{\lambda}' = (\lambda', \mathbf{16})\,, \quad \lambda' &= \left(0,0,1,1,1,1,1,1,\frac{1}{2},1\right)\\
&= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + \frac{7}{2}\alpha_6 + \frac{3}{2}\alpha_7 + \frac{5}{2}\alpha_8\,.
\end{aligned} \tag{8.2}$$

This $\mathfrak{so}(16)$-pack has lowest DE_{18} weight

$$\gamma' = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8\,, \qquad \varpi_{IA}(\gamma') = \lambda'\,. \tag{8.3}$$

Equation (8.2) is easily checked by calculating the energy stored in the flux,

$$e^{2\langle\lambda',\vec{h}\rangle} = \frac{1}{g_s}\frac{M_s^5 R_3 \cdots R_8}{R_1 R_2} M_p^9 R_1 \cdots R_{10} = M_p^{15} R_3^2 \cdots R_8^2 R_9 R_{10}^2\,,$$

where $M_s = M_p^{3/2} R_{10}^{1/2}$ is the string scale, and $g_s = (M_p R_{10})^{3/2}$ is the string coupling constant.

The flux $\mathcal{F}^{\mathrm{RR}}_{345678}$ is associated to

$$\tilde{\lambda}'' = (\lambda'', \mathbf{1}), \qquad \lambda'' = (1,1,0,0,0,0,0,0,1,0)\,.$$

This singlet $\mathfrak{so}(16)$-pack has DE_{18} weight

$$\gamma'' = \beta_{-1} + 2\beta_0 + 2\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 + 2\beta_5 + 2\beta_6 + 2\beta_7 + 2\left(\sum_{j=1}^{6}\beta_{8+j}\right) + \beta_{15} + \beta_{16}, \tag{8.4}$$

(So that $\varpi_{IA}(\gamma'') = \lambda''$.) We conclude that the $\mathfrak{so}(16)$-pack

$$\tilde{\lambda} = (\lambda' + \lambda'', \mathbf{16}) = \left(1,1,1,1,1,1,1,1,\frac{3}{2},1;\mathbf{16}\right).$$

With lowest weight

$$\gamma = \beta_{-1}+2\beta_0+3\beta_1+3\beta_2+3\beta_3+3\beta_4+3\beta_5+3\beta_6+3\beta_7+\beta_8+2\Big(\sum_{j=1}^{6}\beta_{8+j}\Big)+\beta_{15}+\beta_{16}, \tag{8.5}$$

corresponds to the charge that couples to A_0 — the open-string end-point charge. Note that $\gamma^2 = 0$ and that, in fact, γ is $W(DE_{18})$-equivalent to δ_2 from (7.6), and therefore its multiplicity in DE_{18} is

$$\operatorname{mult}\gamma = \operatorname{mult}\delta_2 = 16\,.$$

All 16 weights of $\tilde{\lambda}$ can be obtained from γ by Weyl reflections around $\beta_9, \dots, \beta_{16}$, and therefore square to 0 like γ. As we mentioned in section 7.3, we do not know the physical meaning of the multiplicity.

9. The Orbifold $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$

We will briefly remark on a more complicated orbifold — $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$.

First, let us recall a few of the physical properties of M-theory on $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$. We take the supersymmetric case, without discrete torsion. The $T^4/\mathbb{Z}_2$ singular spaces can be deformed into smooth K_3 spaces. We then get M-theory on $K_3 \times K_3 \times T^2$. According to [24] there must also be 24 M2-branes in this compactification which are points on $K_3 \times K_3$. Alternatively, one can argue [24] that M-theory on K_3 is dual to heterotic string theory on T^3 [61]. The compactification of M-theory on $K_3 \times K_3 \times T^2$ is then dual to heterotic string theory on $K_3 \times T^5$. If the heterotic gauge-bundle is trivial on K_3, then 24 NS5-branes are required to cancel anomalies.

Let us analyze the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on E_{10}. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ charges are determined by the values of $k_0 + k_8$ and $k_4 + k_8$. They can be expressed in terms of the two imaginary roots

$$\tau_1 = (2,2,2,2,2,2,1,1,1,3)\,, \qquad \tau_2 = (2,2,1,1,1,3,2,2,2,2)\,.$$

The subset of roots

$$\Delta^{(\mathrm{inv})} = \{\alpha\colon \quad (\tau_1|\alpha) \equiv (\tau_2|\alpha) \equiv 0 \pmod 2\}\,, \tag{9.1}$$

defines the untwisted sector of M-theory on $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$. The 24 exceptional M2-branes that are required by [24] correspond to the imaginary root

$$\tau_{M2} = (2,2,1,1,1,1,1,1,1,1)\,.$$

It is now interesting to note that

$$(\tau_{M2}|\alpha) \equiv 0 \pmod 2, \qquad \forall\alpha \in \Delta^{(\mathrm{inv})}.$$

In fact,

$$\tau_{M2} \equiv \tau_1 + \tau_2 \pmod 2\,.$$

JHEP06(2005)057

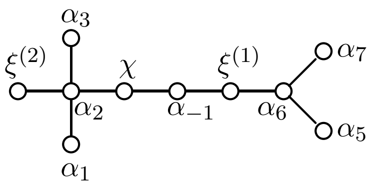

Figure 10: The Dynkin diagram of the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-invariant subalgebra of E_{10} that is associated with the $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$ orbifold.

Let us find the analog of the $DE_{10} \subset E_{10}$ subgroup from section 4. The steps are similar to section 4.3. We set

$$\begin{aligned} \xi^{(1)} &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8 \,, \\ \xi^{(2)} &= \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8 \,, \\ \chi &= \xi^{(1)} + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \,, \end{aligned}$$

and take α_{-1}, α_1, α_3, α_5, α_6, α_7, χ, $\xi^{(1)}$, $\xi^{(2)}$ as a basis. The resulting Dynkin diagram is depicted in figure 10.

Exploring possible analogs of DE_{18} will be left for future work. We have seen in section 6 that the number 8 of extra nodes needed to extend the Dynkin diagram of DE_{10} to DE_{18} was the rank of $\mathfrak{so}(16)$, and 16 was the number of exceptional branes. By the same token, we expect the Kac-Moody algebra that describes the $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$ orbifold to be of rank $10 + (16 + 16 + 24)/2 = 38$, since there are 2×16 exceptional Kaluza-Klein monopoles (corresponding to each separate $T^4/\mathbb{Z}_2$ as in section 3.6) and 24 exceptional M2-branes. It should correspond to a Dynkin diagram with black and white nodes such that the black nodes generate an $\mathfrak{so}(16) + \mathfrak{so}(16) + \mathfrak{so}(24)$ subalgebra, corresponding to the twisted sectors, and the commutant of $\mathfrak{so}(16) + \mathfrak{so}(16) + \mathfrak{so}(24)$ should contain the Kac-Moody algebra depicted in figure 10.

Heterotic string theory on $K_3 \times T^2$ is dual to M-theory on $X \times S^1$, where X is a certain K_3-fibered Calabi-Yau manifold [62, 63]. It follows that if we could understand fully the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of E_{10}, we might be able to give a Lie-algebraic description for M-theory on certain Calabi-Yau manifolds.

10. Conclusions and discussion

In simple $\mathbb{Z}_2$ orbifolds of M-theory on T^{10}, the $\mathbb{Z}_2$ action on spacetime defines a $\mathbb{Z}_2$ action on E_{10}. The spacetime fields and branes of the untwisted sector of the orbifold correspond to roots of the $\mathbb{Z}_2$-invariant subalgebra of E_{10} — the algebra we called $\mathfrak{g}^{(\text{inv})}$. We have seen that this algebra contains a Kac-Moody subalgebra DE_{10}, but $\mathfrak{g}^{(\text{inv})}$ is bigger.

The twisted sectors of the orbifold comprise of branes that correspond to imaginary roots that are orthogonal mod 2 to all the $\mathbb{Z}_2$ invariant roots. Both twisted and untwisted sectors appear to be encoded in a larger structure — the Kac-Moody algebra DE_{18}. This algebra contains an $\mathfrak{so}(16)$ subalgebra whose commutant $\mathfrak{g}^{(\mathrm{com})}$ contains DE_{10}. The infinite dimensional Lie algebra $\mathfrak{g}^{(\mathrm{com})}$ is also larger than DE_{10}, and it would be very interesting to compare it to $\mathfrak{g}^{(\mathrm{inv})}$.

DE_{18} has two distinct Weyl equivalence classes of prime isotropic roots (roots that square to zero and are not a nontrivial multiple of any other root). Roots in the first class have multiplicity 8, while roots in the other class have multiplicity 16. We have given several examples of brane-charges that correspond to roots from each class (see equations (7.7), (7.8), (7.10), (8.5)).

Finally, we constructed a Kac-Moody algebra that describes the untwisted sector of a $(T^4/\mathbb{Z}_2) \times (T^4/\mathbb{Z}_2)$ orbifold. It would be interesting to extend this algebra to include twisted sectors.

Another direction for further study is a possible connection with K-theory. K-theory classifies Ramond-Ramond charges and fluxes in perturbative string theory ([64] and references therein), while real roots of E_{10} and DE_{18} appear to classify fluxes nonperturbatively, and prime isotropic imaginary roots extend the notion of charge. In particular, $\mathbb{Z}_2$ orientifolds [39] have been discussed in the K-theory context in [65, 66]. It would be interesting to understand the various discrete types of orientifolds in terms of DE_{18}; their M-theory interpretation is described in [67]–[69]. It would also be interesting to study the physical interpretations of other Kac-Moody algebras, such as the ones appearing in [70].

Finally, let us discuss our results in the context of the "mysterious duality" between toroidally compactified M-theory and del-Pezzo surfaces [71]. Iqbal, Neitzke and Vafa have established an intriguing correspondence between brane charges of M-theory on T^k ($k = 0, \dots, 8$) and cohomology classes of rational curves in the del-Pezzo surface $\mathbb{B}_k$, which can be described as $\mathbb{CP}^2$ blown up at k points. (See also [72] for a recent discussion.) The connection with E_{10} appears to be that the root lattice of E_k is a sublattice of the $H^2(\mathbb{Z})$ cohomology lattice of $\mathbb{B}_k$. In fact, it appears that the "mysterious duality" can be extended to $k = 10$. The del-Pezzo surface would be replaced by a noncompact Calabi-Yau surface which can be constructed by starting with a $\mathbb{CP}^2$, picking two cubic varieties on it and blowing-up their 9 intersection points, and then removing one cubic (hence the noncompactness). This gives a "$\frac{1}{2}K_3$" surface and the $H^2(\mathbb{Z})$ cohomology lattice is $Q(E_{10})$. (Infinite dimensional Lie algebras have been introduced in the context of this del-Pezzo surface in [73, 74].)

Instantons in M-theory on T^{10} are connected with real roots in $Q(E_{10})$, which in turn are related to cohomology classes of $\frac{1}{2}K_3$ which square to 2. The duality between brane charges and del-Pezzo surfaces with smaller k can be obtained from $\frac{1}{2}K_3$ by blowing down, in a way similar to the one described in [71] for lower k's.

It would be very interesting to study imaginary roots of E_{10} in the context of the generalized Iqbal-Neitzke-Vafa duality. They would correspond to curves of genus 1. Also, the orbifolds of T^{10} that we discussed in section 3 are described by assigning a $\mathbb{Z}_2$-gradation on $Q(E_{10})$. On the del-Pezzo side of the duality, this corresponds to an element of $H^2(\mathbb{Z}_2)$. Perhaps this could be further interpreted as a Stiefel-Whitney class of a real vector bundle.

JHEP06(2005)057

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A. Proof that $DE_{10} \subset \mathfrak{g}^{(\text{inv})}$

We will now complete the details of the proof from section 4.3. Using the map υ, defined in (4.8), we can construct an injective homomorphism of Lie algebras,

$$\tilde{\upsilon} : DE_{10} \to \mathfrak{g}^{(\text{inv})} \subset E_{10}$$

such that

a. For any $\alpha \in \Delta(DE_{10})$ we have $\tilde{\upsilon}(\mathfrak{g}(DE_{10})_\alpha) \subset \mathfrak{g}_{\psi(\alpha)}$, and the restriction of $\tilde{\upsilon}$ to $\mathfrak{g}(DE_{10})_\alpha$ is an injection. [Here $\mathfrak{g}(E_{10})_\alpha$ and $\mathfrak{g}(DE_{10})_\alpha$ are the root spaces of DE_{10} and E_{10}, as defined in (2.1).]

b. $\tilde{\upsilon}$ is an isomorphism between the Cartan subalgebras $\mathfrak{h}(DE_{10})$ and $\mathfrak{h}(E_{10})$.

c. If $\alpha \in \Delta_{\text{Re}}(DE_{10})$ (a real root) then $\tilde{\upsilon}$ is an isomorphism between the root spaces $\mathfrak{g}(DE_{10})_\alpha$ and $\mathfrak{g}_{\psi(\alpha)}$.

Proof. $\tilde{\upsilon}$ can be defined naturally on the Cartan subalgebra $\mathfrak{h}(DE_{10})$. Pick Chevalley generators $e'_i \in \mathfrak{g}(DE_{10})_{\gamma_i}$ $(i = -1 \ldots 8)$, and pick nonzero elements $x_i \in \mathfrak{g}(E_{10})_{\upsilon(\gamma_i)}$. Define $\tilde{\upsilon}(e'_i) = x_i$. The Serre relations among the e'_i's are satisfied by the x_i's. We can see this by using the Weyl-group $W(E_{10})$ to turn pairs of x_i's into simple roots. Using the invariant bilinear forms on DE_{10} and on E_{10} we can find $y_i \in \mathfrak{g}(E_{10})_{-\upsilon(\gamma_i)}$ such that $(x_i|y_j) = \delta_{ij}$. Pick Chevalley generators $f'_i \in \mathfrak{g}(DE_{10})_{-\gamma_i}$ and set $\tilde{\upsilon}(f'_i) = y_i$.

The map $\tilde{\upsilon}$ is well-defined. To see that it is an injection, note that the kernel $\operatorname{Ker}\tilde{\upsilon}$ is an ideal of DE_{10} that intersects $\mathfrak{h}(DE_{10})$ trivially. Therefore, according to [25], $\operatorname{Ker}\tilde{\upsilon} = 0$.

Parts (b) and (c) follow immediately from (a).

B. Proof that $DE_{10} \subset \mathfrak{g}^{(\text{com})}$

We now complete the missing details of proposition 5.1. Set $x^\pm$ to be nonzero generators of the root spaces $\mathfrak{g}(DE_{18})_{\pm\chi}$, and let $\mathfrak{g}' \subset DE_{18}$ be the smallest subalgebra that contains the set

$$\{e_{-1}, f_{-1}, \cdots, e_6, f_6, e_8, f_8, x^+, x^-\} \subset DE_{18(10)}\,. \tag{B.1}$$

We will now show that $\mathfrak{g}' \simeq DE_{10}$. Consider the set of positive real roots $\beta_{-1}, \ldots, \beta_6, \beta_8, \chi$. These roots all square to 2, and their intersection matrix is encoded in the Dynkin diagram of figure 8, which, as we saw, is the Dynkin diagram of DE_{10}. We can therefore construct a surjective map $\phi : DE_{10} \to \mathfrak{g}'$ that maps the Chevalley generators of DE_{10} to the corresponding elements (B.1). We need to prove that ϕ is injective, i.e. that there are no extra relations among the elements of (B.1) in addition to those of the Kac-Moody algebra DE_{10}. To see this note that $\{\phi^{-1}(h_{-1}), \dots, \phi^{-1}(h_8)\}$ generate the Cartan subalgebra $\mathfrak{h}(DE_{10})$, and therefore the kernel of ϕ intersects $\mathfrak{h}(DE_{10})$ trivially. But the kernel of ϕ is an ideal of DE_{10} and a Kac-Moody algebra has no nontrivial ideals that intersect the Cartan subalgebra trivially. (This follows from the construction in §1 of [25].) It follows that ϕ is an isomorphism of algebras and $DE_{10} \simeq \mathfrak{g}'$.

C. Proof of proposition 6.1

proposition 6.1 states that for

$$0 \neq x \in \mathfrak{g}_\gamma \,, \qquad \gamma = \sum_{i=-1}^{8} k_i \beta_i \in \Delta(DE_{18}) \,,$$

$U(\mathfrak{so}(16))x \simeq L_{\mathfrak{so}(16)}(|k_7|\tilde{\tilde{\Lambda}}_9)$. *Proof.* Let e_i, f_i, h_i $(i = -1, \dots, 16)$ be Chevalley generators for DE_{18}. Suppose, without loss of generality, that $k_7 < 0$. The element x is a linear combination of multiple commutators of $f_{-1}, \dots, f_8$ and f_7 appears $|k_7|$ times. Consider a particular commutator

$$z := \underbrace{[\cdots [f_7, \cdots [f_7, \cdots\cdots]]]}_{|k_7| \text{ times}} \,, \tag{C.1}$$

where the other generators that appear in $\cdots$ are from the list $f_{-1}, \dots, f_6, f_8$. $\mathfrak{so}(16)$ commutes with all these generators, and

$$V := U(\mathfrak{so}(16)) f_7 \simeq L(\tilde{\tilde{\Lambda}}_9)$$

is isomorphic to the fundamental representation of $\mathfrak{so}(16)$ (since $e_7, f_7, h_7, e_9, \dots, h_{16}$ generate a finite $\mathfrak{so}(18)$ Lie algebra). Note that $f_7 \in V$ is a lowest-weight vector (a generator of the weight-space of $\tilde{\tilde{\Lambda}}_9$). Let $V^{\otimes |k_7|}$ be the tensor product of $|k_7|$ copies of the fundamental representation **16**. There is a surjective map $g : V^{\otimes |k_7|} \to U(\mathfrak{so}(16))z$ generated by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{|k_7|} \mapsto [\cdots [v_1, \cdots [v_2, \cdots\cdots]]] \,,$$

where the various $(\cdots)$'s are the same series of commutators as appear in the corresponding expression (C.1). This map sends $f_7 \otimes \cdots \otimes f_7$ to z. Now set

$$W := U(\mathfrak{so}(16))(\underbrace{f_7 \otimes \cdots \otimes f_7}_{|k_7| \text{ times}}) \simeq L(|k_7|\tilde{\tilde{\Lambda}}_9) \,.$$

W is isomorphic to the irreducible representation of rank-$|k_7|$ traceless symmetric tensors. g induces a map $g' : W \to U(\mathfrak{so}(16))z$, since $W \subset V^{\otimes |k_7|}$. Also, $f_7 \otimes \cdots \otimes f_7 \in W$, so g' is surjective. Since W is irreducible, g' is an isomorphism. This proves that $U(\mathfrak{so}(16))z$ is isomorphic to $L_{\mathfrak{so}(16)}(|k_7|\tilde{\tilde{\Lambda}}_9)$. It is easy to extend this proof to x, which is a linear combination of expressions like (C.1).

C.1 Equivalence of root lattices

In this section we prove that:

1. The $\mathfrak{g}^{(\mathrm{inv})}$ root lattice is isomorphic to the DE_{10} root lattice.

2. The $\mathfrak{g}^{(\mathrm{com})}$ root lattice is isomorphic to the DE_{10} root lattice.

Proof. We already established an embedding of the DE_{10} root lattices, in section 4.3 using the isometry υ, and in section 5.4 using the isometry σ. We will now show that both υ and σ are actually surjective on positive roots; the surjection on negative roots then follows automatically. Thus we will have shown that υ and σ are surjective embeddings, i.e. isomorphims.

1. Let $\alpha \in \Delta^+(\mathfrak{g}^{(\mathrm{inv})})$. Then

$$\alpha = \sum_{i=-1}^{2} a_i\alpha_i + 2k\alpha_3 + \sum_{j=4}^{8} a_j\alpha_j; \qquad k, a_i, a_j \in \mathbb{N}\,.$$

Using the E_{10} Cartan matrix corresponding to the E_{10} Dynkin diagram (figure 1) we find

$$\begin{aligned}(\alpha|\alpha)_{E_{10}} = 2&\left(\sum_{i=-1}^{2} a_i^2 + 4k^2 + \sum_{j=4}^{8} a_j^2\right) - \\ &- 2\left(\sum_{i=-1}^{1} a_i a_{i+1} + 2ka_2 + 2ka_4 + a_5a_8 + \sum_{j=4}^{6} a_j a_{j+1}\right).\end{aligned}$$

Note that $\alpha \in \Delta^+(\mathfrak{g}^{(\mathrm{inv})})$ implies that $(\alpha|\alpha)_{E_{10}} \in 2\mathbb{Z}$ and is less than or equal to 2. We rewrite α in terms of $\tilde{\upsilon}(DE_{10}) \subset \mathfrak{g}^{(\mathrm{inv})}$ simple roots, and denote by β the resulting element of $\upsilon(Q(DE_{10})) \subset Q(\mathfrak{g}^{(\mathrm{inv})})$. We find

$$\beta = \sum_{i=-1}^{1} a_i\alpha_i + k\xi + (a_2-k)\alpha_2 + (a_4-2k)\alpha_4 + (a_5-2k)\alpha_5 + (a_6-k)\alpha_6 + a_7\alpha_7 + (a_8-k)\alpha_8\,.$$

Next we calculate $(\beta|\beta)_{DE_{10}}$, using the DE_{10} Cartan matrix from section 4.2, and obtain

$$\begin{aligned}(\beta|\beta)_{DE_{10}} = 2&\left(\sum_{i=-1}^{2} a_i^2 + 4k^2 + \sum_{j=4}^{8} a_j^2\right) - \\ &- 2\left(\sum_{i=-1}^{1} a_i a_{i+1} + 2ka_2 + 2ka_4 + a_5a_8 + \sum_{j=4}^{6} a_j a_{j+1}\right),\end{aligned}$$

so $(\beta|\beta)_{DE_{10}} = (\alpha|\alpha)_{E_{10}}$. Since $(\alpha|\alpha)_{E_{10}} \leq 2$ and is even, and since DE_{10} is hyperbolic, [25, proposition 5.10] implies that $\beta \in \Delta(DE_{10})$, and $\upsilon(\beta) = \alpha$ clearly. We want to show that β is also a positive root: if $k > 0$ we are finished, since if one simple root coefficient of a root is positive, then all simple root coefficients must be non-negative. If $k = 0$, it is equally clear that β is a positive root, by assumption on α. Thus we have proven that $\upsilon : \Delta^+(DE_{10}) \to \Delta^+(\mathfrak{g}^{(\mathrm{inv})})$ is a surjective map.

2. Let $\alpha \in \Delta^+(\mathfrak{g}^{(\mathrm{com})})$. Then

$$\alpha = \sum_{i=-1}^{6} a_i\beta_i + a_8\beta_8 + 2k\beta_7 + \sum_{j=9}^{14} 2k\beta_j + k\beta_{15} + k\beta_{16}\,; \qquad k, a_i \in \mathbb{N}\,.$$

Using the DE_{18} Cartan matrix from (5.1), we find

$$(\alpha|\alpha)_{DE_{18}} = 2\left(\sum_{i=-1}^{6} a_i^2 + a_8^2 + 2k^2\right) - 2\left(\sum_{j=-1}^{5} a_j a_{j+1} + 2ka_6 + a_1 a_8\right).$$

We rewrite α in terms of the $\phi(DE_{10}) \subset \mathfrak{g}^{(\mathrm{com})}$ simple roots, and denote by β the resulting element of $\phi(Q(DE_{10})) \subset Q(\mathfrak{g}^{(\mathrm{com})})$. We find,

$$\beta = \sum_{i=-1}^{5} a_i\beta_i + a_8\beta_8 + k\chi + (a_6 - k)\beta_6.$$

Using the $\mathfrak{g}' \cong DE_{10}$ Dynkin diagram (figure 8), we calculate

$$(\beta|\beta)_{DE_{10}} = 2(\sum_{i=-1}^{6} a_i^2 + a_8^2 + 2k^2) - 2(\sum_{j=-1}^{5} a_j a_{j+1} + 2ka_6 + a_1 a_8)\,,$$

so $(\beta|\beta)_{DE_{10}} = (\alpha|\alpha)_{DE_{18}}$. We remark here that since $\alpha \in \Delta^+(\mathfrak{g}^{(\mathrm{com})})$, then $(\alpha|\alpha)_{DE_{18}} \leq 2$ (see Proposition 5.2c of [25]); however, the converse statement no longer holds, because DE_{18} is not hyperbolic. But we do not need the converse statement. Since DE_{10} is hyperbolic, Proposition 5.10 of [25] implies that $\beta \in \Delta(DE_{10})$, and $\sigma(\beta) = \alpha$ clearly. The same reasoning as in part (1) now shows that $\beta \in \Delta^+(DE_{10})$, so $\sigma : \Delta^+(DE_{10}) \to \Delta^+(\mathfrak{g}^{(\mathrm{com})})$ is a surjective map.

D. Denominator formula for $\mathfrak{g}^{(\mathrm{inv})}$

In this section we will present a "denominator formula" that captures the multiplicities of $\mathfrak{g}^{(\mathrm{inv})}$ roots. Recall the *denominator identity* (formula (10.4.4) of [25]),

$$\prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} (\mathrm{sgn}\, w) e^{w(\rho)-\rho}\,,$$

Here, as usual in character formulas, we expand each side in a formal power series in the formal variables $e^{-\alpha_i}$. The multiplication is according to the rule $e^{-\alpha}e^{-\beta} = e^{-\alpha-\beta}$. The sum on the right-hand side is over all Weyl-group elements w, and $\mathrm{sgn}\, w$ is the "signature" of w — $(+1)$ if w is a product of an even number of simple reflections and (-1) otherwise.

JHEP06(2005)057

The weight ρ is chosen such that $(\rho|\alpha) = 1$ for all simple roots α. For E_{10}, with the assignment of simple roots as in the figure 1, we have

$$\rho = -30\alpha_{-1} - 61\alpha_0 - 93\alpha_1 - 126\alpha_2 - 160\alpha_3 - 195\alpha_4 - 231\alpha_5 - 153\alpha_6 - 76\alpha_7 - 115\alpha_8 \,.$$

We now calculate

$$\prod_{\alpha\in\Delta^+} (1 - e^{-t\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} (\mathrm{sgn}\, w) e^{tw(\rho)-t\rho}$$

and

$$\prod_{\alpha\in\Delta^+} (1 - (-1)^{(\alpha|\tau)} e^{-\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} (\mathrm{sgn}\, w)(-1)^{(\tau|w(\rho)-\rho)} e^{w(\rho)-\rho}\,.$$

It follows that

$$\prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha} \prod_{\alpha\in\Delta^+\backslash\Delta_+^{(\mathrm{inv})}} (1 + e^{-\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} (\mathrm{sgn}\, w)(-1)^{(\tau|w(\rho)-\rho)} e^{w(\rho)-\rho}\,.$$

Multiplying by the original denominator formula, we get

$$\prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} (1 - e^{-\alpha})^{2\,\mathrm{mult}\,\alpha} \prod_{\alpha\in\Delta^+\backslash\Delta_+^{(\mathrm{inv})}} (1 - e^{-2\alpha})^{\mathrm{mult}\,\alpha} =$$

$$= \left(\sum_{w\in W} (\mathrm{sgn}\, w)(-1)^{(\tau|w(\rho)-\rho)} e^{w(\rho)-\rho}\right) \left(\sum_{w\in W} (\mathrm{sgn}\, w) e^{w(\rho)-\rho}\right).$$

Finally, dividing by

$$\prod_{\alpha\in\Delta^+} (1 - e^{-2\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} (\mathrm{sgn}\, w) e^{2w(\rho)-2\rho}$$

we obtain the requisite formula

$$\prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} \left(\tanh\frac{\alpha}{2}\right)^{\mathrm{mult}\,\alpha} = \frac{\left(\sum_{w\in W} (\mathrm{sgn}\, w)(-1)^{(\tau|w(\rho)-\rho)} e^{w(\rho)-\rho}\right)\left(\sum_{w\in W} (\mathrm{sgn}\, w) e^{w(\rho)-\rho}\right)}{\sum_{w\in W} (\mathrm{sgn}\, w) e^{2w(\rho)-2\rho}} \tag{D.1}$$

where we used

$$\prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} \left(\tanh\frac{\alpha}{2}\right)^{\mathrm{mult}\,\alpha} = \prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} \left(\frac{1-e^{-\alpha}}{1+e^{-\alpha}}\right)^{\mathrm{mult}\,\alpha} = \prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} \frac{(1-e^{-\alpha})^{2\,\mathrm{mult}\,\alpha}}{(1-e^{-2\alpha})^{\mathrm{mult}\,\alpha}}\,.$$

Now we can compare (D.1) to a similar expression for DE_{10}. Using similar manipulations we find

$$\prod_{\alpha\in\Delta_+^{(\mathrm{inv})}} \left(\tanh\frac{\alpha}{2}\right)^{\mathrm{mult}'\,\alpha} = \frac{\left(\sum_{w'\in W(DE_{10})} (\mathrm{sgn}\, w') e^{w'(\rho')-\rho'}\right)^2}{\sum_{w'\in W(DE_{10})} (\mathrm{sgn}\, w') e^{2w'(\rho')-2\rho'}}, \qquad \rho' \equiv \rho(DE_{10})\,, \tag{D.2}$$

where mult' denotes DE_{10} multiplicities, and we used the identification $\Delta_+^{(\mathrm{inv})} = \Delta^+(DE_{10})$, where $\mathfrak{h}(DE_{10})^*$ is implicitly identified with $\mathfrak{h}(E_{10})^*$ using (4.8).

Using (2.5) and (4.5) we can write

$$\rho' = \rho + 8\tau' ,$$

where τ' is some root such that

$$\tau - \tau' \in 2Q(E_{10})$$

so that

$$(-1)^{(\alpha|\tau)} = (-1)^{(\alpha|\tau')} \qquad \text{for all } \alpha \in Q(E_{10}).$$

Using (D.2) we get

$$\prod_{\alpha\in\Delta_+^{(\text{inv})}} \left(\tanh\frac{\alpha}{2}\right)^{\text{mult}'\,\alpha} =$$

$$= \frac{\left(\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{w'(\rho)-\rho+8w'(\tau')-8\tau'}\right)^2}{\sum_{w'\in W(DE_{10})}(\operatorname{sgn} w')e^{2w'(\rho)-2\rho+16w'(\tau')-16\tau'}}$$

$$\prod_{\alpha\in\Delta_+^{(\text{inv})}} \left(\tanh\frac{\alpha}{2}\right)^{\text{mult}\,\alpha} =$$

$$= \frac{\left(\sum_{w\in W(E_{10})}(\operatorname{sgn} w)(-1)^{\langle\tau',w(\rho)-\rho\rangle}e^{w(\rho)-\rho}\right)\left(\sum_{w\in W(E_{10})}(\operatorname{sgn} w)e^{w(\rho)-\rho}\right)}{\sum_{w\in W(E_{10})}(\operatorname{sgn} w)e^{2w(\rho)-2\rho}}$$

Let us now discuss the relation between the Weyl groups $W(DE_{10})$ and $W(E_{10})$. We can identify both $W(DE_{10})$ and $W(E_{10})$ as subgroups of the isometry group of the dual of the Cartan subalgebra $\mathfrak{h}(DE_{10})^* \simeq \mathfrak{h}(E_{10})^*$.

Lemma D.1. *We have:*

a. $W(DE_{10}) \subset W(E_{10})$*;*

b. $W(DE_{10}) \simeq \{w \in W(E_{10}): \quad w(\tau) - \tau \in 2Q(E_{10})\}$*;*

c. *The coset* $W(DE_{10})/W(E_{10})$ *is finite and has* 527 *elements.*

Proof. $W(DE_{10})$ is generated by reflections around $\gamma_{-1}, \ldots, \gamma_8$. Using (4.8), we can thus identify $W(DE_{10})$ with the subset of $W(E_{10})$ that is generated by reflections around the real roots $\upsilon(\gamma_{-1}), \ldots, \upsilon(\gamma_8)$. This proves part (a).

To prove (b), consider a fundamental Weyl reflection $r_i \in W(DE_{10})$ around the simple root γ_i. This maps to a reflection $r_{\upsilon(\gamma_i)} \in W(E_{10})$. We now calculate

$$r_{\upsilon(\gamma_i)}(\tau) - \tau = (\tau|\upsilon(\gamma_i))\upsilon(\gamma_i) \in 2Q(E_{10}) ,$$

since $(\tau|\upsilon(\gamma_i))$ is even. $W(E_{10})$ is generated by the fundamental reflections r_i, and it therefore follows that $W(DE_{10}) \subset \{w \in W(E_{10}): \quad w(\tau) - \tau \in 2Q(E_{10})\}$.

To prove $\supset$, take $w \in W(E_{10})$ such that

$$w(\tau) - \tau = 2\beta \qquad \text{for some } \beta \in Q(E_{10}).$$

note first that

$$(\tau|\tau) = \tau^2 = 0 \equiv 0 \pmod 2$$

and therefore $\tau \in Q^{(\mathrm{inv})} \simeq Q(DE_{10})$. According to Proposition 5.10b of [25], since both E_{10} and DE_{10} are hyperbolic, the Weyl groups $W(DE_{10})$ and $W(E_{10})$ are equivalent to a $\mathbb{Z}/2\mathbb{Z}$[1] quotient of the group of automorphisms of the respective root lattices $Q(DE_{10})$ and $Q(E_{10})$. The $\mathbb{Z}/2\mathbb{Z}$ quotient is the identification of the automorphisms $\phi, -\phi \in \mathrm{Aut}(Q)$. From part (a) it follows that every automorphism of $Q(DE_{10})$ can be extended to an automorphism of $Q(E_{10})$. Thus

$$q(w(\alpha)) \equiv (w(\alpha)|\tau) = (\alpha|w^{-1}(\tau)) = (\alpha|\tau - 2w^{-1}(\beta)) \equiv (\alpha|\tau) \equiv q(\alpha)$$
$$\text{for any } \alpha \in Q(DE_{10}) \simeq Q^{(\mathrm{inv})} \subset Q(E_{10}).$$

This proves that w preserves $Q^{(\mathrm{inv})} \simeq Q(DE_{10})$ and therefore

$$W(DE_{10}) \supset \{w \in W(E_{10}): \quad w(\tau) - \tau \in 2Q(E_{10})\}\,.$$

To prove (c), note that if $w_1, w_2 \in W(E_{10})$ then $w_1(\tau) - w_2(\tau) \in 2Q(E_{10})$, if and only if $w_1^{-1}w_2(\tau) - \tau \in 2Q(E_{10})$, and so, by (b), $w_1^{-1}w_2 \in W(DE_{10})$. Therefore, the map $t : W(E_{10})/W(DE_{10}) \to Q(E_{10})/2Q(E_{10})$ that sends the equivalence class of $w \in W(E_{10})$ to the equivalence class of $w(\tau) - \tau$ is an injection. This proves the finiteness of $W(E_{10})/W(DE_{10})$.

To count the size of $W(E_{10})/W(DE_{10})$ we need to calculate the size of the image of t. This is most conveniently done in the basis (2.4). Define the set

$$S{:=}\{(n_1,\ldots,n_{10}): \quad n_i \in \mathbb{Z}\,, \quad \sum_{i=1}^{10} n_i \in 3\mathbb{Z}\,, \qquad \sum_{i=1}^{10}(-1)^{n_i} \in \{-8,-6,0,2,8\}\}$$

Using (2.2), we see that S can be identified with a certain a subset of $Q(E_{10})$. Using (2.4), it is easy to check that the fundamental Weyl reflections preserve $S \subset Q(E_{10})$. S is therefore $W(E_{10})$-invariant. Using the explicit expressions found in section 3, we see that $\tau \in S$. The coset $S/2Q(E_{10})$ [where we use the explicit representation of $Q(E_{10})$ as in (2.4)] contains

$$\binom{10}{9} + \binom{10}{8} + \binom{10}{5} + \binom{10}{4} + \binom{10}{1} = 527\,\text{elements.}$$

Therefore, the image in $Q(E_{10})/2Q(E_{10})$ of the map t contains at most 527 elements. To see that it contains exactly 527 elements, recall that by Proposition 5.7 of [25] all positive prime isotropic roots of E_{10} are $W(E_{10})$-equivalent to τ. Below is a list of such roots $\alpha = w(\tau)$ [in the basis (2.4)] such that $t(w) = [\alpha - \tau]$ (the equivalence class of $\alpha - \tau$ in $Q(E_{10})/2Q(E_{10})$) exhaust all 527 possibilities.

$$\binom{10}{9} = 10 \text{ distinct permutations of} \quad (0,1,1,1,1,1,1,1,1,1),$$

[1]Here we use the notation $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$ to avoid any confusion with the $\mathbb{Z}_2$ orbifold group.

$$\binom{10}{8} = 45 \text{ distinct permutations of } \quad (2,2,1,1,1,1,1,1,1,1) \equiv (0,0,1,1,1,1,1,1,1,1),$$
$$\binom{10}{5} = 252 \text{ distinct permutations of } \quad (2,2,2,2,2,1,1,1,1,1) \equiv (0,0,0,0,0,1,1,1,1,1),$$
$$\binom{10}{4} = 210 \text{ distinct permutations of } \quad (2,2,2,2,2,2,1,1,1,3) \equiv (0,0,0,0,0,0,1,1,1,1),$$
$$\binom{10}{1} = 10 \text{ distinct permutations of } \quad (2,2,2,2,2,2,2,2,4,1) \equiv (0,0,0,0,0,0,0,0,0,1).$$

The $\equiv$'s are $\pmod 2$.

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JHEP06(2005)057

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JHEP06(2005)057